

Oppgave 1 Løs initialverdiproblemet

$$y''(t) - 4y'(t) + 3y(t) = \delta(t - 5), \quad y(0) = y'(0) = 1,$$

der δ er deltafunksjonen.

Oppgave 2 La funksjonen f være definert ved $f(x) = \cos(x)$ for $0 < x < \pi$.

a) Finn Fourier-sinusrekken til $f(x)$.

b) Skisser summen av Fourier-sinusrekken til $f(x)$ på intervallet $[-2\pi, 2\pi]$.

Finn verdien til Fourier-sinusrekken til $f(x)$ i punktene $x = -\pi/4$, $x = 0$ og $x = \pi/2$.

Oppgave 3 La C være sirkelen $\{z \in \mathbb{C} : |z - 2| = 2\}$ orientert mot klokka.

Finn verdien av linjeintegralet

$$\oint_C \frac{1}{(z-1)(z-7)} dz.$$

Oppgave 4 Vis ved hjelp av Cauchy-Riemannligningene at

$$f(z) = ze^{iz}$$

er en hel funksjon, dvs. at $f(z)$ er analytisk i hele \mathbb{C} .

Oppgave 5 La $R > 0$ og S_R være halvsirkelen med parametrisering

$$z(\theta) = Re^{i\theta}, \quad 0 \leq \theta \leq \pi.$$

La $x \geq 0$ og bruk ML-ulikheten til å vise at

$$\lim_{R \rightarrow \infty} \int_{S_R} \frac{1}{4 + w^4} e^{iwx} dw = 0.$$

Oppgave 6

- a) Finn og klassifiser de singulære punktene til funksjonen

$$f(z) = \frac{z^{4n} - 1}{z^{2n+1}}, \quad n = 1, 2, \dots$$

Finn Laurentrekken til $f(z)$ om $z = 0$ og regn ut residyet i $z = 0$.

- b) Bruk residyregning for å vise at

$$\int_0^{2\pi} \cos(n\theta) \sin(n\theta) d\theta = 0, \quad n = 1, 2, \dots$$

Oppgave 7

- a) Finn alle løsninger på formen $u(x, t) = F(x)G(t)$ som tilfredsstiller den partielle differensialligningen

$$u_t(x, t) + (1 + t^2)(2u(x, t) - u_{xx}(x, t)) = 0, \quad x \in [0, \frac{\pi}{2}], \quad t > 0, \quad (1)$$

og randbetingelsene

$$u(0, t) = u_x(\frac{\pi}{2}, t) = 0, \quad t \geq 0. \quad (2)$$

- b) Finn en løsning som tilfredsstiller (1) og (2) og i tillegg initialbetingelsen

$$u(x, 0) = \sin(3x) + \sin(17x), \quad x \in [0, \frac{\pi}{2}].$$

Oppgave 1 Løs initialverdiproblemet

$$y''(t) - 4y'(t) + 3y(t) = \delta(t - 5), \quad y(0) = y'(0) = 1,$$

der δ er deltafunksjonen.

By applying Laplace transform to the equation,
we have

$$\underbrace{\mathcal{L}[y'']}_{(s^2 Y(s) - s - 1)} - 4 \underbrace{\mathcal{L}[y']}_{(s Y(s) - 1)} + 3Y(s) = e^{-5s}$$

Thus

$$\underline{(s^2 - 4s + 3)} Y(s) = (s - 3) + e^{-5s}$$

hence

$$(s-3)(s-1)$$

$$Y(s) = \frac{1}{s-1} + \frac{e^{-5s}}{(s-3)(s-1)}$$

By partial fraction,

$$\frac{1}{(s-3)(s-1)} = \frac{A}{(s-3)} + \frac{B}{(s-1)}$$

$$\Rightarrow A + B = 0$$

$$-A - 3B = 1$$

$$\Rightarrow A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

therefore,

$$Y(s) = \frac{1}{s-1} + e^{-5s} \left(\frac{1}{s-3} - \frac{1}{s-1} \right) \frac{1}{2}$$

Now, we want to find the inverse Laplace transform of $Y(s)$.

Using t -shifting, we have

$$\mathcal{L}^{-1} \left[e^{-as} \cdot \frac{1}{s-b} \right] = u(t-a) e^{b(t-a)}$$

and

$$y(t) = \mathcal{L}^{-1} [Y(s)] = e^t + \frac{1}{2} u(t-5) \left(e^{3(t-5)} - e^{(t-5)} \right)$$

Remark)

i) standard procedure: 1. Apply Laplace transform to the eq.

2. Solve $Y(s)$

3. Find $y(t) = \mathcal{L}^{-1} [Y(s)]$

Oppgave 2 La funksjonen f være definert ved $f(x) = \cos(x)$ for $0 < x < \pi$.

a) Finn Fourier-sinusrekken til $f(x)$.

b) Skisser summen av Fourier-sinusrekken til $f(x)$ på intervallet $[-2\pi, 2\pi]$.

Finn verdien til Fourier-sinusrekken til $f(x)$ i punktene $x = -\pi/4$, $x = 0$ og $x = \pi/2$.

a) Fourier sine series of $f(x)$ gives

$2L$ odd extension of $f(x)$.

$\frac{\pi}{\pi}$

$$f_1(-x) = -f_1(x)$$

we define

$$f_1(x) = \begin{cases} f(x) = \cos(x), & 0 < x < \pi. \\ -f(-x) = -\cos(x), & -\pi < x < 0. \end{cases}$$

Fourier sine series $S_{f_1}(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

$$\text{and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(x) \cdot \sin(n\pi x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(n\pi x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} (\sin(n-1)x + \sin(n+1)x) dx$$

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \sin(\beta) \cos(\alpha)$$

$$= \begin{cases} \frac{1}{\pi} \left[\frac{-\cos((n+1)x)}{n+1} \right]_0^{\pi} = 0 & (n=1) \end{cases}$$

$$\begin{cases} \frac{1}{\pi} \left[\frac{-\cos((n-1)x)}{n-1} - \frac{\cos((n+1)x)}{n+1} \right]_0^{\pi} & (n \neq 1) \end{cases}$$

$$= \frac{1}{\pi} \left(\frac{1}{n-1} \underbrace{(-\cos((n-1)\pi) + 1)}_{\substack{\parallel \\ (-1)^n}} \right) + \frac{1}{n+1} \left(\underbrace{-\cos((n+1)\pi) + 1}_{\substack{\parallel \\ (-1)^{n+2} = (-1)^2 (-1)^n = (-1)^n}} \right)$$

$$= \begin{cases} 0, & n: \text{odd} \\ \frac{2}{\pi} \cdot \frac{2n}{n^2-1}, & n: \text{even.} \end{cases}$$

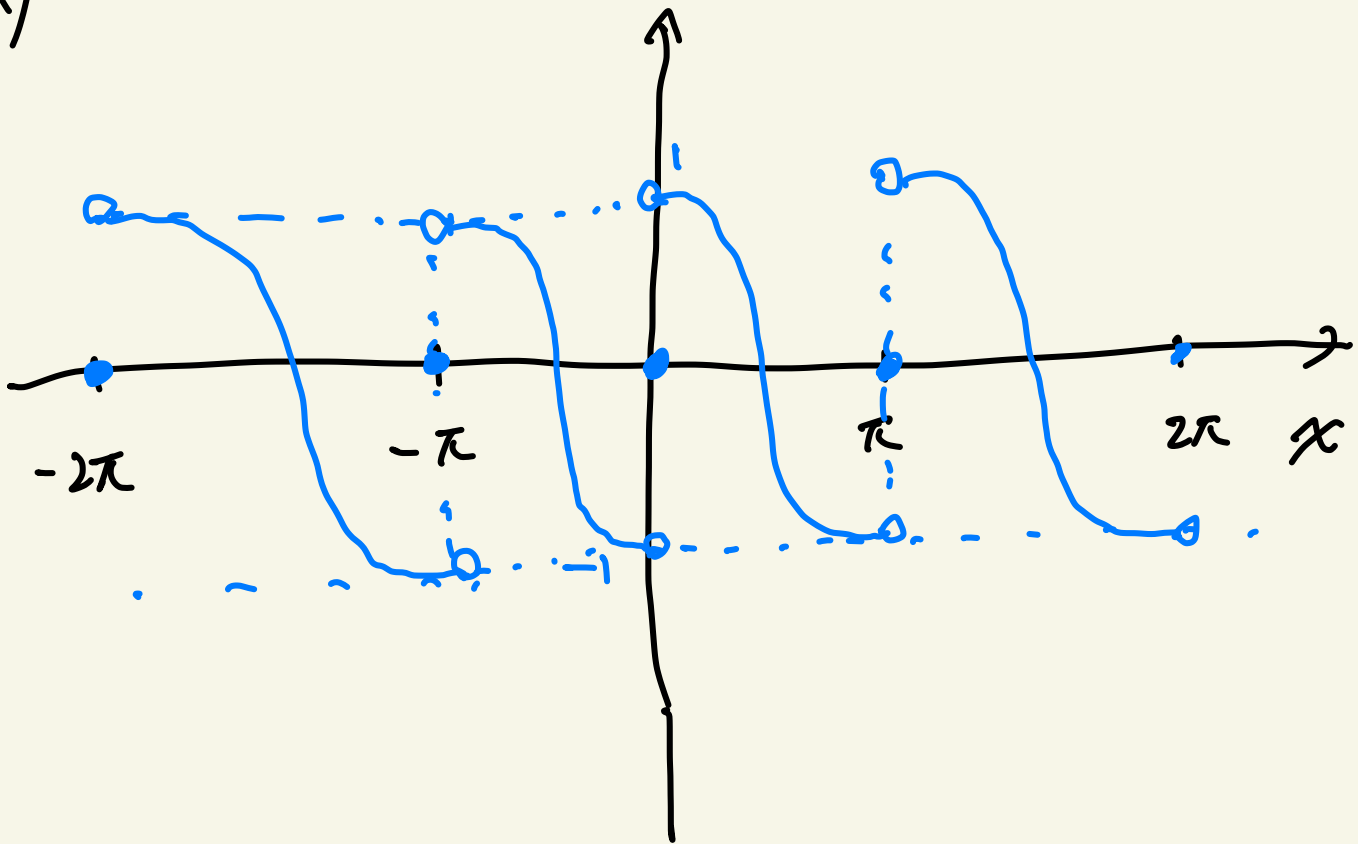
Thus, the Fourier sine series

$$S_{f_1}(x) = \sum_{\substack{n=1 \\ n: \text{even}}}^{\infty} \frac{4}{\pi} \cdot \frac{n}{n^2-1} \sin(n\pi) = \sum_{m=1}^{\infty} \frac{4}{\pi} \frac{2m}{(2m)^2-1} \times \sin(2m\pi)$$

$2m=n$

h)

$$f(-x) = -f(x)$$



$$S_f\left(-\frac{\pi}{4}\right) = -f\left(-\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

$$S_f(0) = \frac{f_1(0^+) + f_1(0^-)}{2} = \frac{1 + (-1)}{2} = 0.$$

$$S_f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

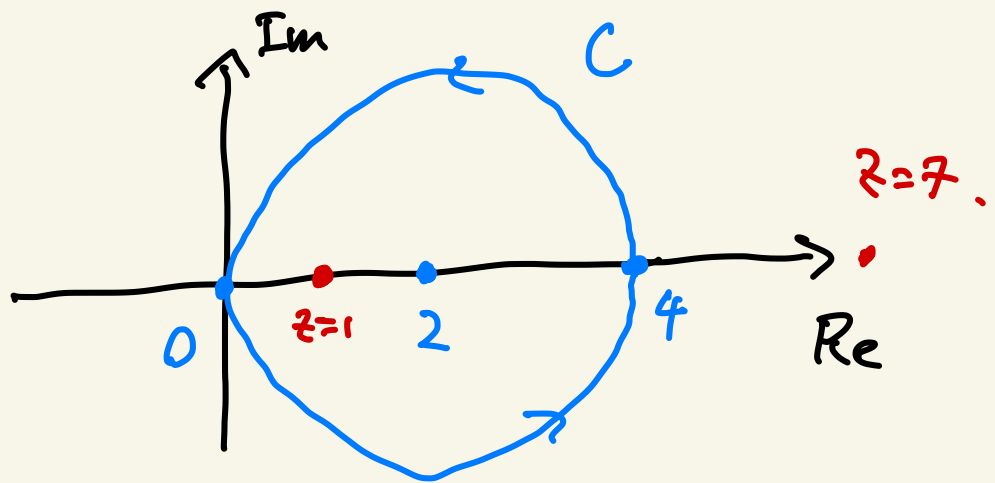
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Oppgave 3 La C være sirkelen $\{z \in \mathbb{C} : |z - 2| = 2\}$ orientert mot klokka.

Finn verdien av linjeintegralet

$$\oint_C \frac{1}{(z-1)(z-7)} dz.$$

Let $f(z) = \frac{1}{(z-1)(z-7)}$. $f(z)$ has singular points at $z=1$, and $z=7$.



The curve C encloses only one singularity $z=1$. Thus by the residue theorem, we have

$$\begin{aligned} \oint_C f(z) dz &= 2\pi i \operatorname{Res}(f(z))_{z=1} = 2\pi i \lim_{z \rightarrow 1} \left(\frac{1}{z-7} \right) \\ &= -\frac{\pi i}{3} \end{aligned}$$

→

Oppgave 4 Vis ved hjelp av Cauchy-Riemannligningene at

$$f(z) = ze^{iz}$$

er en hel funksjon, dvs. at $f(z)$ er analytisk i hele \mathbb{C} .

Cauchy - Riemann eq. tells $f(x,y) = U(x,y) + iV(x,y)$
if

satisfies $\begin{cases} U_x = V_y, \\ U_y = -V_x, \end{cases}$ then f is analytic.

First we need to find U and V .

$$\begin{aligned} f(z) &= f(x+iy) = (x+iy) e^{i(x+iy)} = e^{-y} (x+iy) e^{ix} \\ &= e^{-y} (x+iy) (\cos(x) + i\sin(x)) \\ &= \underbrace{e^{-y} (x\cos(x) - y\sin(x))}_{U(x,y)} + i \underbrace{e^{-y} (x\sin(x) + y\cos(x))}_{V(x,y)} \end{aligned}$$

Now we check the Cauchy - Riemann equations.

$$U_x(x,y) = e^{-y} (\cos(x) - x\sin(x) - y\cos(x))$$

$$U_y(x, y) = e^{-y} (-x \cos(x) + y \sin(x) - \sin(x))$$

$$V_x(x, y) = e^{-y} (-y \sin(x) + \sin(x) + x \cos(x)) = -U_y$$

for all x, y .

$$V_y(x, y) = e^{-y} (\cos(x) - y \cos(x) - x \sin(x)) = U_x$$

for all x, y .

Thus, Cauchy-Riemann equations hold for

all $z = x + iy \in \mathbb{C}$. This implies $f(z)$ is analytic everywhere in \mathbb{C} .

Oppgave 5 La $R > 0$ og S_R være halvsirkelen med parametrisering

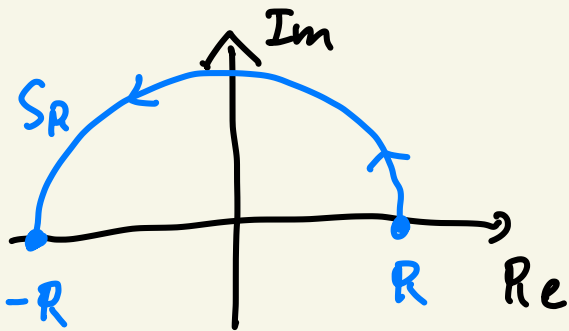
$$z(\theta) = Re^{i\theta}, \quad 0 \leq \theta \leq \pi.$$

La $x \geq 0$ og bruk ML-ulikheten til å vise at

$$\lim_{R \rightarrow \infty} \int_{S_R} \frac{1}{4+w^4} e^{iwx} dw = 0.$$

Let $f(w) = \frac{1}{4+w^4}$. XZO.

Let $w = u+iv$. If $w \in S_R$, then $v \geq 0$



$$\text{Then } |e^{iwx}| = |e^{i(u+iv)x}| = |e^{-vx} \cdot e^{iux}|$$

$$\leq e^{-vx} \cdot |e^{iux}| \leq 1$$

$$\begin{matrix} \leq \\ (v \geq 0) \\ (x \geq 0) \end{matrix}$$



$$\text{Thus, } \max_{w \in S_R} \left| \frac{1}{4+w^4} e^{iwx} \right| \leq \max_{w \in S_R} \left| \frac{1}{4+w^4} \right|$$

$$(|a+b| \geq |a| - |b|)$$

$$\leq \max_{w \in S_R} \frac{1}{|w|^4 - 4} = \frac{1}{R^4 - 4} \cdot \frac{1}{|w^4 + 4|}$$

$|w^4 + 4| \geq |w|^4 - 4$

Then, by using ML-inequality,

$$\left| \int_{S_R} \frac{1}{4 + w^4} e^{iwx} dx \right| \leq \max_{w \in S_R} \left| \frac{1}{4 + w^4} e^{iwx} \right| \cdot \pi R$$

$$\leq \frac{1}{R^4 - 4} \cdot \pi R \xrightarrow{R \rightarrow \infty} 0.$$

Thus, we proved $\lim_{R \rightarrow \infty} \int_{S_R} \frac{1}{4 + w^4} e^{iwx} dx = 0$
for $x \geq 0$.

Opgave 6

a) Finn og klassifiser de singulære punktene til funksjonen

$$f(z) = \frac{z^{4n} - 1}{z^{2n+1}}, \quad n = 1, 2, \dots$$

Finn Laurentrekken til $f(z)$ om $z = 0$ og regn ut residyet i $z = 0$.

b) Bruk residyregning for å vise at

$$\int_0^{2\pi} \cos(n\theta) \sin(n\theta) d\theta = 0, \quad n = 1, 2, \dots$$

$$a) f(z) = \frac{z^{4n} - 1}{z^{2n+1}} = z^{2n-1} - \frac{1}{z^{2n+1}}$$

$f(z)$ has singular point at $z=0$,
and this is $2n+1$ order pole.

$f(z) = z^{2n-1} - \frac{1}{z^{2n+1}}$ is already the

Laurent series of the form

$$\sum_{\substack{m \\ m \neq 0}}^{\infty} a_m z^m + \sum_{\substack{m \\ m}}^{\infty} \frac{b_m}{z^m}, \quad a_{2n-1} = 1 \quad \text{otherwise,} \\ b_{2n+1} = -1. \quad a_m = 0 \\ b_m = 0$$

(consider $f(z) = z^2$ and you want)

(to compute the Taylor series of $f(z)$ at $z=0$,
 $f(z)$ is already a Taylor series
around $z=0$!
around $z=0$.)

Thus, $z^{2n-1} - \frac{1}{z^{2n+1}}$ is the Laurent series
of $f(z)$ around $z=0$.

Residue of $f(z)$ at $z=0$.

$\text{Res}_{z=0}(f(z)) = b_1 = 0$, since the Laurent series
tells that we don't have $\frac{b_1}{z}$ in the series.

