Oppgave 1 Løs initialverdiproblemet

$$y''(t) - 4y'(t) + 3y(t) = \delta(t-5), \quad y(0) = y'(0) = 1,$$

der δ er deltafunksjonen.

Oppgave 2 La funksjonen f være definert ved $f(x) = \cos(x)$ for $0 < x < \pi$.

a) Finn Fourier-sinusrekken til $f(x)$.

b) Skisser summen av Fourier-sinusrekken til $f(x)$ på intervallet $[-2\pi, 2\pi]$. Finn verdien til Fourier-sinusrekken til $f(x)$ i punktene $x = -\pi/4$, $x = 0$ og $x = \pi/2$.

Oppgave 3 La C være sirkelen $\{z \in \mathbb{C} : |z-2| = 2\}$ orientert mot klokka.

Finn verdien av linjeintegralet

$$\oint_C \frac{1}{(z-1)(z-7)}\, dz.$$

Oppgave 4 Vis ved hjelp av Cauchy-Riemannligningene at

$$f(z) = ze^{iz}$$

er en hel funksjon, dvs. at $f(z)$ er analytisk i hele $\mathbb{C}$.

Oppgave 5 La $R > 0$ og S_R være halvsirkelen med parametrisering

$$z(\theta) = Re^{i\theta}, \quad 0 \le \theta \le \pi.$$

La $x \ge 0$ og bruk ML-ulikheten til å vise at

$$\lim_{R\to\infty} \int_{S_R} \frac{1}{4+w^4} e^{iwx}\, dw = 0.$$

$$f(z) = \frac{z^{4n} - 1}{z^{2n+1}}, \quad n = 1, 2, \ldots .$$

Finn Laurentrekken til $f(z)$ om $z = 0$ og regn ut residyet i $z = 0$.

b) Bruk residyregning for å vise at

$$\int_0^{2\pi} \cos(n\theta)\sin(n\theta)\, d\theta = 0, \quad n = 1, 2, \ldots .$$

Oppgave 7

a) Finn alle løsninger på formen $u(x,t) = F(x)G(t)$ som tilfredsstiller den partielle differensialligningen

$$u_t(x,t) + (1 + t^2)(2u(x,t) - u_{xx}(x,t)) = 0, \quad x \in [0, \frac{\pi}{2}], \quad t > 0, \qquad (1)$$

og randbetingelsene

$$u(0,t) = u_x(\frac{\pi}{2}, t) = 0, \quad t \geq 0. \qquad (2)$$

b) Finn en løsning som tilfredsstiller (1) og (2) og i tillegg initialbetingelsen

$$u(x,0) = \sin(3x) + \sin(17x), \quad x \in [0, \frac{\pi}{2}].$$

Oppgave 1 Løs initialverdiproblemet

$$y''(t) - 4y'(t) + 3y(t) = \delta(t-5), \quad y(0) = y'(0) = 1,$$

der δ er deltafunksjonen.

By applying Laplace transform for the equation, we have

$$\underbrace{(s^2 Y(s) - s - 1)}_{\mathcal{L}[y'']} - 4\underbrace{(sY(s) - 1)}_{\mathcal{L}[y']} + 3Y(s) = e^{-5s}$$

Thus

$$\underbrace{(s^2 - 4s + 3)}_{(s-3)(s-1)} Y(s) = (s-3) + e^{-5s}$$

hence

$$Y(s) = \frac{1}{s-1} + \frac{e^{-5s}}{(s-3)(s-1)}$$

By partial fraction,

$$\frac{1}{(s-3)(s-1)} = \frac{A}{(s-3)} + \frac{B}{(s-1)}$$

$$\Rightarrow \begin{matrix} A + B = 0 \\ -A - 3B = 1 \end{matrix} \Rightarrow \begin{matrix} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{matrix}$$

therefore,

$$Y(s) = \frac{1}{s-1} + e^{-5s} \cdot \left(\frac{1}{s-3} - \frac{1}{s-1}\right)\frac{1}{2}$$

Now. we want to find the inverse Laplace transform of $Y(s)$.

Using t-Shifting, we have

$$\mathcal{L}^{-1}\left[e^{-as} \cdot \frac{1}{s-b}\right] = u(t-a)\, e^{b(t-a)}$$

and

$$y(t) = \mathcal{L}^{-1}[Y(s)] = e^{t} + \frac{1}{2}u(t-5)\left(e^{3(t-5)} - e^{(t-5)}\right)$$

//

Remark)

i) standard procedure : 1. Apply Laplace transform to the eq.

2. Solve $Y(s)$

3. Find $y(t) = \mathcal{L}^{-1}[Y(s)]$

Oppgave 2 La funksjonen f være definert ved $f(x) = \cos(x)$ for $\boxed{0 < x < \pi}$.

a) Finn Fourier-sinusrekken til $f(x)$.

b) Skisser summen av Fourier-sinusrekken til $f(x)$ på intervallet $[-2\pi, 2\pi]$.
Finn verdien til Fourier-sinusrekken til $f(x)$ i punktene $x = -\pi/4$, $x = 0$ og $x = \pi/2$.

a) Fourier sine series of $f(x)$ gives

$\underset{\pi}{2\underline{L}}$ odd extension of $f(x)$.

$$f_1(-x) = -f_1(x)$$

we define

$$f_1(x) = \begin{cases} f(x) = \cos(x), & 0 < x < \pi. \\ -f(-x) = -\cos(x), & -\pi < x < 0. \end{cases}$$

Fourier sine series $S_{f_1}(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_1(x) \cdot \sin(nx)\,dx$

$$= \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(nx)\,dx$$

$$\underset{\varphi}{=} \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\sin((n-1)x) + \sin((n+1)x)\right) dx$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$$

$$= \begin{cases} \dfrac{1}{\pi}\cdot\left[\dfrac{-\cos((n+1)x)}{\underset{\overset{\shortparallel}{2}}{n+1}}\right]_0^{\pi} = 0 & (n=1) \\[2ex] \dfrac{1}{\pi}\left[\dfrac{-\cos((n-1)x)}{n-1} - \dfrac{\cos((n+1)x)}{n+1}\right]_0^{\pi} & (n \neq 1) \end{cases}$$

$$= \frac{1}{\pi}\left(\frac{1}{n-1}\Big(\underbrace{-\cos((n-1)\pi)}_{\overset{\shortparallel}{(-1)^n}} + 1\Big) + \frac{1}{n+1}\Big(\underbrace{-\cos((n+1)\pi)}_{\overset{\shortparallel}{(-1)^{n+2}}} + 1\Big)\right)$$

$$(-1)^{n+2} = (-1)^2(-1)^n = (-1)^n$$

$$= \begin{cases} 0, & n: \text{odd} \\[2ex] \dfrac{2}{\pi}\cdot\dfrac{2n}{n^2-1}, & n: \text{even}. \end{cases}$$

(n: positive integer)

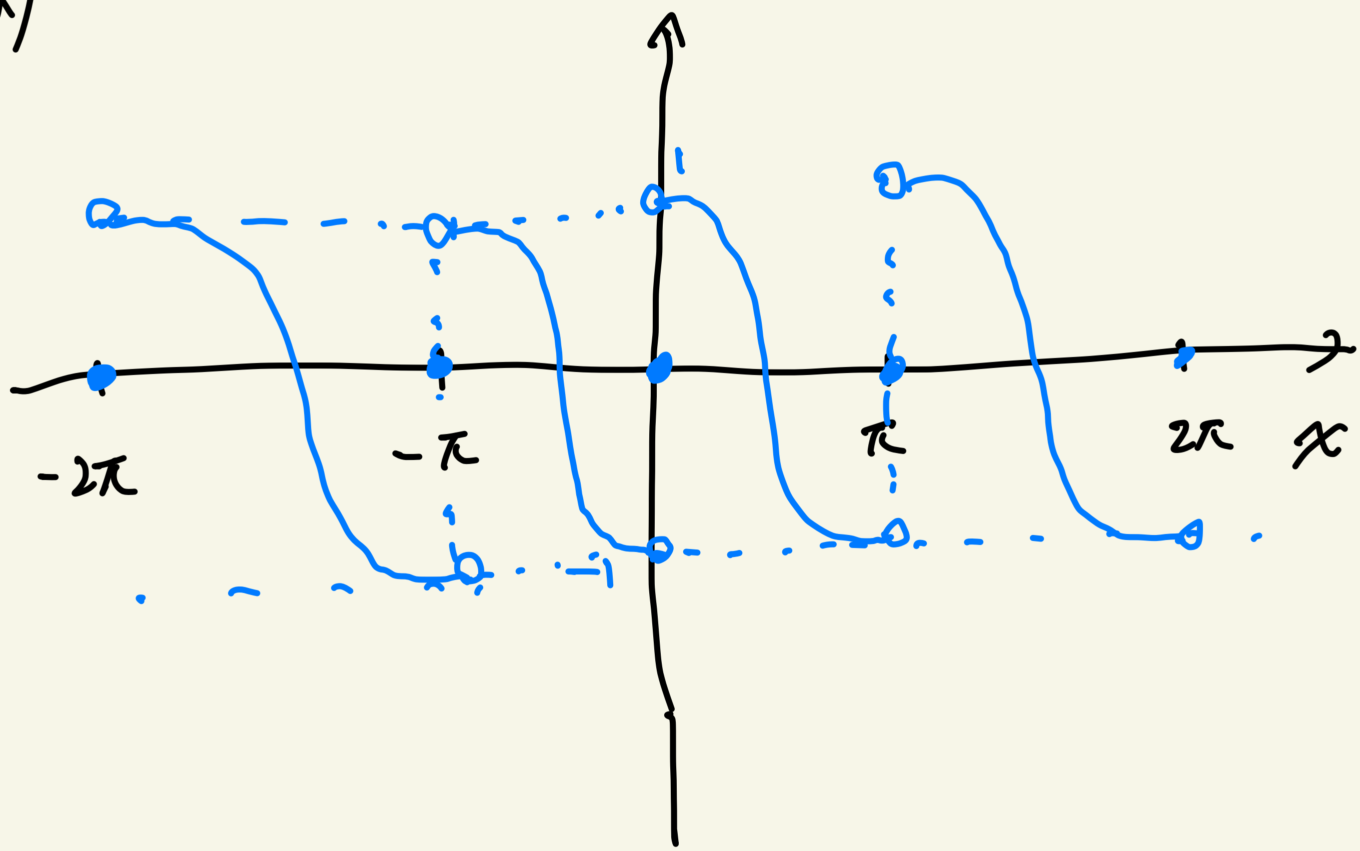
Thus, the Fourier sine series

$$S_{f_1}(x) = \sum_{\substack{n=1 \\ n:\text{even}}}^{\infty} \frac{4}{\pi}\cdot\frac{n}{n^2-1}\sin(nx) = \sum_{m=1}^{\infty} \frac{4}{\pi}\,\frac{2m}{(2m)^2-1} \times \sin(2mx)$$

$2m = n$

h)

$f(-x) = -f(x)$



$$S_f\left(-\frac{\pi}{4}\right) = -f\left(-\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

$$S_f(0) = \frac{f_1(0^+) + f_1(0^-)}{2} = \frac{1+(-1)}{2} = 0.$$

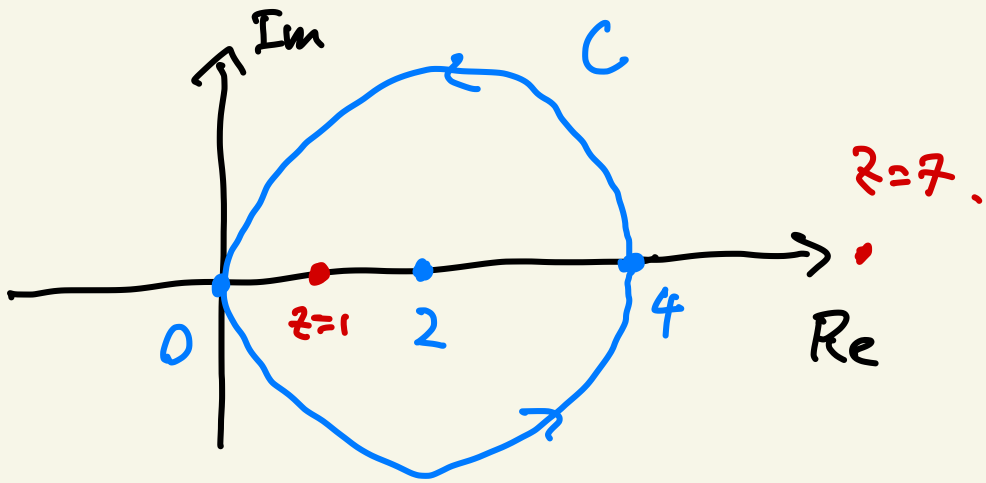
$$S_f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

Oppgave 3 La C være sirkelen $\{z \in \mathbb{C} : |z-2| = 2\}$ orientert mot klokka.

Finn verdien av linjeintegralet

$$\oint_C \frac{1}{(z-1)(z-7)}\,dz.$$

Let $f(z) = \dfrac{1}{(z-1)(z-7)}$. $f(z)$ has singular points at $z=1$, and $z=7$.



The curve C encloses only one singularity $z=1$.
Thus by the residue theorem, we have

$$\oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=1}(f(z)) = 2\pi i \lim_{z\to 1}\left(\frac{1}{z-7}\right)$$

$$= -\frac{\pi i}{3}$$

$\#$

Oppgave 4 Vis ved hjelp av Cauchy-Riemannligningene at

$$f(z) = ze^{iz}$$

er en hel funksjon, dvs. at $f(z)$ er analytisk i hele $\mathbb{C}$.

Cauchy-Riemann eq. tells if $f(x,y) = u(x,y) + iv(x,y)$

satisfies $\begin{cases} u_x = v_y, \\ u_y = -v_x, \end{cases}$ then f is analytic.

First we need to find u and v.

$$f(z) = f(x+iy) = (x+iy)e^{i(x+iy)} = e^{-y}(x+iy)e^{ix}$$

$$= e^{-y}(x+iy)(\cos(x) + i\sin(x))$$

$$= \underbrace{e^{-y}(x\cos(x) - y\sin(x))}_{u(x,y)} + i\underbrace{e^{-y}(x\sin(x) + y\cos(x))}_{v(x,y)}.$$

Now we check the Cauchy-Riemann equations.

$$u_x(x,y) = e^{-y}(\cos(x) - x\sin(x) - y\cos(x))$$

$$u_y(x,y) = e^{-y}(-x\cos(x) + y\sin(x) - \sin(x))$$

$$v_x(x,y) = e^{-y}(-y\sin(x) + \sin(x) + x\cos(x)) = -u_y$$

for all x, y.

$$v_y(x,y) = e^{-y}(\cos(x) - y\cos(x) - x\sin(x)) = u_x$$

for all x, y.

Thus, Cauchy-Riemann equations hold for all $z = x+iy \in \mathbb{C}$. This implies $f(z)$ is analytic everywhere in $\mathbb{C}$.

Oppgave 5 La $R > 0$ og S_R være halvsirkelen med parametrisering

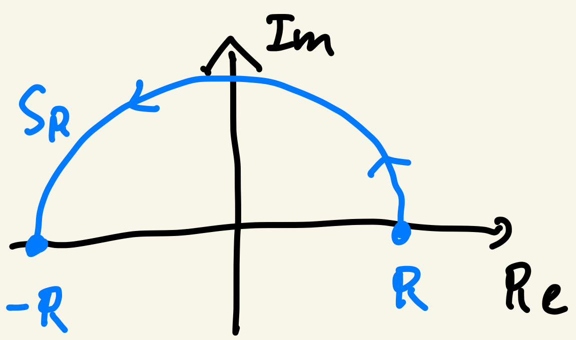
$$z(\theta) = Re^{i\theta}, \quad 0 \leq \theta \leq \pi.$$

La $x \geq 0$ og bruk ML-ulikheten til å vise at

$$\lim_{R\to\infty} \int_{S_R} \frac{1}{4+w^4} e^{iwx}\, dw = 0.$$

Let $f(w) = \dfrac{1}{4+w^4}$. $x \geq 0$.

Let $w = u+iv$. If $w \in S_R$, then $v \geq 0$



Then $\left|e^{iwx}\right| = \left|e^{i(u+iv)x}\right| = \left|e^{-vx} \cdot e^{iux}\right|$

$\leq e^{-vx} \cdot \left|e^{iux}\right| \leq 1$

$\begin{pmatrix} v \geq 0 \\ x \geq 0 \end{pmatrix}$

Thus, $\max_{w\in S_R} \left|\dfrac{1}{4+w^4} e^{iwx}\right| \leq \max_{w\in S_R} \left|\dfrac{1}{4+w^4}\right|$

$(|a+b| \geq |a| - |b|)$

$$\leq \max_{w \in S_R} \frac{1}{|w|^4-4} = \frac{1}{R^4-4}.$$

$$\frac{1}{|w^4+4|}$$

$$|w^4+4| \geq |w|^4-4$$

Then, by using ML-inequality,

$$\left| \int_{S_R} \frac{1}{4+w^4} e^{iwx} \, dx \right| \leq \max_{w \in S_R} \left| \frac{1}{\underset{4}{1+w^4}} e^{iwx} \right| \cdot \pi R$$

$$\leq \frac{1}{R^4-4} \cdot \pi R \xrightarrow{R \to 0} 0.$$

Thus, we proved $\displaystyle \lim_{R \to \infty} \int_{S_R} \frac{1}{\underset{4}{1+w^4}} e^{iwx} \, dx = 0$ for $x \geq 0$.

Oppgave 6

a) Finn og klassifiser de singulære punktene til funksjonen

$$f(z) = \frac{z^{4n}-1}{z^{2n+1}}, \quad n = 1, 2, \ldots.$$

Finn Laurentrekken til $f(z)$ om $z = 0$ og regn ut residyet i $z = 0$.

b) Bruk residyregning for å vise at

$$\int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = 0, \quad n = 1, 2, \ldots.$$

a) $f(z) = \dfrac{z^{4n}-1}{z^{2n+1}} = z^{2n-1} - \dfrac{1}{z^{2n+1}}.$

$f(z)$ has singular point at $z=0$, and this is $2n+1$ order pole.

$f(z) = z^{2n-1} - \dfrac{1}{z^{2n+1}}$ is already the Laurent series of the form

$$\sum_{m=0}^{\infty} a_m z^m + \sum_{m=1}^{\infty} \frac{b_m}{z^m}, \quad a_{2n-1} = 1, \quad b_{2n+1} = -1.$$

otherwise, $a_m = 0$, $b_m = 0$

(consider $f(z) = z^2$ and you want)

(to compute the Taylor series of $f(z)$ at $z=0$,
$f(z)$ is already a Taylor series around $z=0$!)
z^2

Thus. $z^{2n-1} - \frac{1}{z^{2n+1}}$ is the Laurent series of $f(z)$ around $z=0$.

Residue of $f(z)$ at $z=0$.

$\operatorname{Res}_{z=0}(f(z)) = b_1 = 0$, since the Laurent series tells that we don't have $\frac{b_1}{z}$ in the series.

h) $I = \int_0^{2\pi} \cos(n\theta)\sin(n\theta)\, d\theta$.

Let $z = e^{i\theta}$, then $dz = ie^{i\theta}d\theta = iz\,d\theta$.

Then, $\cos(n\theta) = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) = \frac{1}{2}(z^n + \frac{1}{z^n})$

$\sin(n\theta) = \frac{1}{2i}(e^{in\theta} - e^{-in\theta}) = \frac{1}{2i}(z^n - z^{-n})$

This gives us

$$I = \oint_{C:|z|=1} \frac{1}{4i}\left(z^n + \frac{1}{z^n}\right)\left(z^n - \frac{1}{z^n}\right)\frac{dz}{iz}$$

counter clock-wise

$$= \oint_C -\frac{1}{4z}\left(z^{2n} - \frac{1}{z^{2n}}\right) dz$$

$$= -\frac{1}{4}\oint_C f(z)\,dz,$$ where $f(z)$ is the function from (a).

By using the residue theorem

$$I = -\frac{1}{4}\cdot 2\pi i \operatorname*{Res}_{z=0} f(z) = -\frac{\pi i}{2}\cdot 0 = 0 \;\#$$

Oppgave 7

a) Finn alle løsninger på formen $u(x,t) = F(x)G(t)$ som tilfredsstiller den partielle differensialligningen

$$u_t(x,t) + (1+t^2)(2u(x,t) - u_{xx}(x,t)) = 0, \quad x \in [0, \frac{\pi}{2}], \quad t > 0, \quad (1)$$

og randbetingelsene

$$u(0,t) = u_x(\frac{\pi}{2}, t) = 0, \quad t \geq 0. \quad (2)$$

b) Finn en løsning som tilfredsstiller (1) og (2) og i tillegg initialbetingelsen

$$u(x,0) = \sin(3x) + \sin(17x), \quad x \in [0, \frac{\pi}{2}].$$

Let $u(x,t) = F(x)G(t)$. Then (1) becomes

$$F(x)G'(t) + (1+t^2)(2F(x)G(t) - F''(x)G(t)) = 0$$

We seek for non-zero solutions, and dividing by $(1+t^2)F(x)G(t)$ gives us

$$\frac{G'(t)}{(1+t^2)G(t)} = \frac{F''(x) - 2F(x)}{F(x)}$$

The left hand side depends only on t and the right hand side depends only on x.

This equality must hold for all x and t,
this means both hand sides are constant.

$$\frac{G'(t)}{(1+t^2)G(t)} = \frac{F''(x) - 2F(x)}{F(x)} = k. \quad (\text{const.})$$

This gives us two ODEs:

$$F''(x) = (k+2)F(x) \quad \text{—— } (3)$$

and

$$G'(t) = k(1+t^2)G(t) \quad \text{—— } (4)$$

The boundary condition (2):

$$u(0,t) = u_x\left(\frac{\pi}{2}, t\right) = 0 \quad , t \geq 0$$

which means $F(0)G(t) = F'\left(\frac{\pi}{2}\right)G(t) = 0$

So $F(0) = F'\left(\frac{\pi}{2}\right)$ —— $(2)'$

since we want non-zero solution

$(G(t) = 0 \Rightarrow u(x,t) = 0)$.

We solve (3) and $(2)'$.

If $k+2>0$, then

$$F(x)=Ae^{\sqrt{k+2}\,x}+Be^{-\sqrt{k+2}\,x}$$

$$F(0)=0 \Rightarrow A+B=0$$

$$F'\left(\frac{\pi}{2}\right)=0 \Rightarrow \sqrt{k+2}\,A e^{\sqrt{k+2}\frac{\pi}{2}}-\sqrt{k+2}\,B e^{-\sqrt{k+2}\frac{\pi}{2}}$$

$$=\sqrt{k+2}\,A\left(e^{\sqrt{k+2}\frac{\pi}{2}}-e^{-\sqrt{k+2}\frac{\pi}{2}}\right)=0$$

$$\Rightarrow A=0.$$

So we only get zero-solution in this case.

If $k+2=0$

$$F(x)=Ax+B.$$

$$F(0)=0 \Rightarrow B=0$$

$$F'\left(\frac{\pi}{2}\right)=0 \Rightarrow A=0$$

So we only get zero solution in this case.

If $k+2<0$, then $\underset{\text{let}}{\wedge}$ $k+2=-p^2$ $\left(p=\sqrt{-k-2}\right)$

$$F(x) = A\cos(px) + B\sin(px)$$

$$F(0) = 0 \Rightarrow A = 0$$

$$F'\left(\frac{\pi}{2}\right) = 0 \Rightarrow Bp\cos\left(p\frac{\pi}{2}\right) = 0$$

$$\Rightarrow p = 2n+1. \qquad (n \in \mathbb{Z})$$

$$\Rightarrow k = -(2n+1)^2 - 2$$

Thus, $F_n(x) = \sin((2n+1)x)$

solves the ODE (3) and (2)'.

$$\int \frac{G'(t)}{G(t)}\,dt = k\int (1+t^2)\,dt$$

$$\log(G(t)) + C = k\left(\frac{t^3}{3} + t\right) + C'$$

Thus, ODE (4) has a solution for $k+2<0$

$$G_n(t) = C_0 e^{k\left(\frac{t^3}{3}+t\right)} = C_0 e^{(-(2n+1)^2-2)\left(\frac{t^3}{3}+t\right)}.$$

As a conclusion, $\underbrace{F_n(x)G_n(t)}_{u_n(x,t)=} = C\sin((2n+1)x) \times e^{(-(2n+1)^2-2)(\frac{t^3}{3}+t)}$

solves the PDE (1) and (2).

$u_n(x,t)$ solves (1) and (2) for all $n \in \cancel{\mathbb{Z}}$.

Because the PDE ((1) and (2)) is linear and homogeneous, the superposition of u_n also solves the same problem.

The superposition is given by

$$u(x,t) = \sum_{n=0}^{\infty} A_n u_n(x,t)$$

$$= \sum_{n=0}^{\infty} A_n \sin((2n+1)x)\, e^{(-(2n+1)^2-2)(\frac{t^3}{3}+t)} \quad //$$

(☆)

(A_n can be any constant depending n).

h) Solving the problem with

$$u(x,0) = \sin(3x) + \sin(17x). \quad \text{— (5)}$$

We want to satisfy (5) by choosing A_n from the general solution (☆).

From (☆),

$$u(x,0) = \sum_{n=0}^{\infty} A_n \sin((2n+1)x) \underset{(5)}{=} \sin(3x) + \sin(17x).$$

This means, $A_1 = 1$, $A_8 = 1$, and $A_n = 0$ $(n \neq 1, n \neq 8)$.

Thus,

$$u(x,t) = \sin(3x) e^{-11\left(\frac{t^3}{3}+t\right)} + \sin(17x) e^{-291\left(\frac{t^3}{3}+t\right)}$$

solves the problem.

$\#$

<u>Remark</u>: Standard procedure!! (separation of variables)

(i) Assume $u(x,t) = F(x)G(t)$.

(ii) Derive ODEs for F and G.

Derive boundary condition for F.

(iii) Solve the ODE for F (F_n) depending on k.

(iv) Find the Solution G_n which corresponds to F_n

Finding non-zero solution $u_n(x,t) = F_n(x) G_n(t)$

(v) Since the Problem (PDE) is linear and homogeneous, superposition of u_n also solve the problem.

(see "Theorem 2" in Lecture 9)

$$u(x,t) = \sum_{n=0}^{\infty} A_n F_n(x) G_n(t)$$

Find A_n's such that the initial condition is satisfied.

Complex Fourier Series

Let a function f defined on $[-L, L]$,
then the Fourier series of $f(x)$ is

$$S_f(x) = \underbrace{a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)}_{\text{"Fourier series"}} \quad \text{— ①}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi n x}{L}}}_{\text{"complex Fourier series"}} \quad \text{— ②}$$

$$e^{\frac{i\pi n x}{L}} = \cos\left(\frac{n\pi x}{L}\right) + i \sin\left(\frac{n\pi x}{L}\right).$$

$a_0 = c_0$.

and $n \neq 0$.

③

$$\begin{cases} c_n + c_{-n} = a_n \\ i c_n - i c_{-n} = b_n \end{cases} \Rightarrow \begin{cases} c_n = \frac{1}{2}(a_n - i b_n) \\ c_{-n} = \frac{1}{2}(a_n + i b_n) \end{cases}$$

As far as $f(x)$ is real valued function, $[-L, L] \longrightarrow \mathbb{R}$, a_n and b_n in ① are always real. Then, from ③, c_n and c_{-n} are complex conjugate ($c_n = \overline{c_{-n}}$).

In this setting,

① and ② are just same.

PDEs.

$u_t(x,t) = \cdots\cdots$

If $-\infty < x < \infty$,

think of Fourier transform!

Step 1), Apply the Fourier transform for the eq.

$$\hat{u}_t(w, t) = \cdots\cdots$$

Step 2) Solve the above eq. for t.

$$\hat{u}(w, t) = \cdots\cdots$$

Step 3) Find the solution

$$u(x,t) = \mathcal{F}^{-1}[\hat{u}(w,t)] = \cdots\cdots$$

If $a \le x \le b$

think of separation of variables !

standard procedure ....