Mathematics 4K (TMA4120)

Lecture 7

Summary: Fourier series

1. Fourier series of $p = 2L$-periodic $f(x)$:

$$\boxed{S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)}$$

where $\quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx,$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

2. Odd and even functions

Even $\quad g(-x) = g(x) \qquad \boxed{\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx}$

Odd $\quad h(-x) = -h(x) \qquad \boxed{\int_{-L}^{L} h(x)dx = 0}$

even $\cdot$ even $=$ odd $\cdot$ odd $=$ even; odd $\cdot$ even $=$ odd

Summary: Fourier series

4. Fourier $\sin$ and $\cos$ series:

f even: $S_f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$,

$$\boxed{a_0 = \frac{1}{L}\int_0^L f(x)\,dx}, \quad \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}$$

f odd: $S_f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$,

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$$

5. Even and odd $2L$-periodic extensions of $f(x)$, $0 \le x \le L$:

$f_1(x)$, $x \in \mathbb{R}$, even, $2L$-periodic, $f_1 = f$ on $[0, L]$,

$S_{f_1}(x) = \cos$-series $=:$ the Fourier $\cos$ series of f

$f_2(x)$, $x \in \mathbb{R}$, odd, $2L$-periodic, $f_2 = f$ on $[0, L]$,

$S_{f_2}(x) = \sin$-series $=:$ the Fourier $\sin$ series of f

6. **OBS:** $f = f_1 = f_2$ on $[0, L]$, $\quad f_1 \neq f_2$, $\quad f$ not defined on $[0, L]^c$!

Lecture 7: Fourier Series

Kreyszig: Section 11.4 (10th ed.) and Section 11.4 in 9th ed.!

1. Approximation with trigonometric polynomials
2. Bessel's inequality, Parseval's identity
3. Complex Fourier series
4. Examples

Section 11.4 in 9th ed. available on course wiki page (Fremdriftsplan).

Homework: Repeat complex *numbers, absolute values, exponentials* [Mat 3]

Lecture 7

Approximation

Taylor: $f \in C^{k+1} \Rightarrow f(x) \approx \tilde{a}_0 + \tilde{a}_1 x + \cdots + \tilde{a}_k x^k$
around $x=0$.

$$\tilde{a}_k = \frac{f^{(k)}(0)}{k!}$$

Fourier: $f(x)$: 2π-Periodic

$$S_f(x) = \lim_{k\to\infty} \underbrace{\left(a_0 + \sum_{n=1}^{k} (a_n \cos(nx) + b_n \sin(nx))\right)}_{\parallel \atop S_{f,k}}$$

("Fourier partial sum")

we want to approximate $f(x)$ by

$$f(x) \approx S_{f,k}(x)$$

Definition

a) $P_k(x)$ is called a trigonometric polynomial of degree k, if

$$P_k(x) = A_0 + \sum_{n=1}^{k} \left(A_n \cos(nx) + B_n \sin(nx) \right)$$

where A_0, A_n, B_n can be chosen arbitrarily.

b) L^2 "mean-square" error

$$\|f-g\|^2 := \int_{-\pi}^{\pi} |f(x)-g(x)|^2 \, dx$$

notice $\|f\|^2 = \langle f, f \rangle = \int_{-\pi}^{\pi} |f(x)|^2 dx.$

Theorem 1

a) $\|f(x) - S_{f,k}(x)\|^2 \leq \|f(x) - P_k(x)\|^2$

for all P_k.

($S_{f,k}$ is the best possible approximation) of f, amongst all k-th degree trigonometric polynomials

h) $$\| f(x) - S_{f,k}(x) \|^2 = \int_{-\pi}^{\pi} (f(x))^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{k} [a_n^2 + b_n^2] \right]$$

Proof)

Step 1). $\| f(x) - P_k(x) \|^2$

$$= \int_{-\pi}^{\pi} (f(x))^2 - \underbrace{2 f(x) P_k(x)}_{\text{Step 2}} + \underbrace{(P_k(x))^2}_{\text{Step 3}} dx$$

Step 2) $\int_{-\pi}^{\pi} f(x) P_k(x) dx$

$$= A_0 \underbrace{\int_{-\pi}^{\pi} f(x) dx}_{2\pi a_0} + \sum_{n=1}^{k} \Big[A_n \underbrace{\int_{-\pi}^{\pi} f(x) \cos(nx) dx}_{\pi a_n}$$

$$+ B_n \int_{-\pi}^{\pi} f(x) \sin(nx) dx \Big]$$

$\pi \; b_n$

Step 3) $\displaystyle\int_{-\pi}^{\pi} (P_k(x))^2 dx \underset{\uparrow}{=} \pi \left[2A_0^2 + \sum_{n=1}^{k} (A_n^2 + B_n^2) \right]$

(using orthogonality of trigonometric system

$f, g \in \mathcal{T} \Rightarrow \displaystyle\int_{-\pi}^{\pi} f \cdot g \, dx = 0$

$(f \neq g)$)

Step 4) From step 1,

$P_k^2 = \left(A_0 + \sum_{n=1}^{k} A_n \cos(nx) + B_n \sin(nx) \right)^2$

$$\|f - P_k\|^2 = \int_{-\pi}^{\pi} (f(x))^2 dx$$

$$-2\pi \left[2A_0 a_0 + \sum_{n=1}^{k} [A_n a_n + B_n b_n] \right]$$

$$+\pi \left[A_0^2 + \sum_{n=1}^{k} (A_n^2 + B_n^2) \right]$$

$$\underset{\uparrow}{=} \int_{-\pi}^{\pi} (f(x))^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{k} (a_n^2 + b_n^2) \right]$$

$(A-a)^2 = A^2 - 2aA + a^2$

$\Leftrightarrow A^2 - 2aA = -a^2 + (A-a)^2$

$$+\pi \left[2(a_0 - A_0)^2 + \sum_{n=1}^{k=1} \left[(a_n - A_n)^2 + (b_n - B_n)^2 \right] \right]$$

(— parts 2 ○)

Step 5). To minimize this, choose $A_n = a_n$
$B_n = b_n$

$$\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} (f(x))^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}[a_n^2 + b_n^2]\right]$$

$$\leq \|f - P_k\|^2 \quad \text{for all } P_k$$

$\blacksquare$

Ex 1.

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1. & 0 < x < \pi. \end{cases}$$

$f(x+2\pi) = f(x)$, for all $x \in \mathbb{R}$.

$$S_f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\cdots\right)$$

Problem: find the minimum k and

the corresponding $S_{f,k}$ such that

$$\| f - S_{f,k} \|^2 \le \frac{1}{15} \| f \|^2$$

("relative L^2 error" $\le \frac{1}{15}$)

$$\| f \|^2 = \int_{-\pi}^{\pi} (f(x))^2 dx = 2\pi$$

$$\frac{2\pi}{15} \approx 0.419 \ldots \ldots$$

<u>$k=1$</u>. $S_{f,1}(x) = \frac{4}{\pi} \sin(x)$

$$\| f - S_{f,1} \|^2 \underset{\text{thm 1.4)}}{=} \int_{-\pi}^{\pi} (f(x))^2 dx - \pi \cdot \left(\frac{4}{\pi}\right)^2$$

$$= 1.2 \ldots \quad > 0.419 \ldots$$

<u>$k=5$</u>. $\| f - S_{f,5} \|^2 = (2\pi) - \pi \cdot \left(\frac{4}{\pi}\right)^2 \left(1^2 + \frac{1}{3^2} + \frac{1}{5^2}\right)$

$$= 0.4206 \ldots > 0.419 \ldots$$

$k=7$

$$\|f - S_{f,7}\|^2 = (2\pi) - \pi\cdot\left(\frac{\cancel{8}\ 16}{\pi^2}\right)\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}\right)$$

$$= 0.316 \quad < 0.419.$$

$$S_{f,7}(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)\right).$$

minimum $k=7$ //.

Bessel and Parseval

(Thm 1.ck)

$$\|f(x) - S_{f,k}(x)\|^2 = \int_{-\pi}^{\pi} (f(x))^2\,dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2+b_n^2)\right]$$

$$\to 2a_0^2 + \sum_{n=1}^{k}(a_n^2+b_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx$$

$k \to \infty$

[Bessel's inequality]

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx$$

[Parseval's identity]

If f satisfies $\int_{-\pi}^{\pi} (f(x))^2 dx < \infty$, then

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx$$

Proof) "Ikke Pensum"

proving $\lim_{k\to\infty} \int_{-\pi}^{\pi} (f(x) - S_{f,k}(x))^2 dx = 0.$

Ex2. Calculate.

$$Q = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$$

We use

$$f(x) = \begin{cases} -1, & -\pi < x < 0. \\ 1, & 0 < x < \pi. \end{cases}$$

$$S_f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \cdots\cdots\right)$$

by using Parseval's identity.

$$\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx}_{\substack{\shortparallel \\ 2.}} \overset{\downarrow}{=} \left(\frac{4}{\pi}\right)^2\left(1^2 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right)$$

$$= \left(\frac{4}{\pi}\right)^2 \cdot Q$$

Thus, $Q = \frac{\pi^2}{16} \cdot 2 = \frac{\pi^2}{8}$ //.

Complex Fourier Series.

Recall from previous courses ("Mat 3"?)...

$z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$. $i^2 = -1$.

$$(1)\quad e^{z} = e^{x+iy} = e^{x}\left(\cos(y) + i\sin(y)\right)$$

$$\Downarrow$$

$$(2)\quad e^{iy} + e^{-iy} = 2\cos(y)$$

$$(3)\quad e^{iy} - e^{-iy} = 2i\sin(y).$$

Fourier series with $p = 2\pi$.

$$S_f(x) = a_0 + \sum_{n=1}^{\infty}\left(\underline{a_n\cos(nx) + b_n\sin(nx)}\right)$$

$$a_n\cos(nx) + b_n\sin(nx)$$

$$\overset{(2),(3)}{=} a_n\cdot\frac{1}{2}\left(e^{inx} + e^{-inx}\right) + b_n\,\frac{1}{2i}\left(e^{iy} - e^{-iy}\right)$$

$$\underset{\frac{1}{i}=-i}{=} \underbrace{\frac{1}{2}\left(a_n - ib_n\right)}_{C_n} e^{inx} + \underbrace{\frac{1}{2}\left(a_n + ib_n\right)}_{C_{-n}} e^{-inx}$$

Thus,

$$S_{f,k}(x) = a_0 + \sum_{n=1}^{k}\left(C_n e^{inx} + C_{-n} e^{-inx}\right)$$

$$\overset{c_0 := a_0}{=} \sum_{n=-k}^{k}\left(C_n e^{inx}\right) \quad \text{— } \bigstar$$

Remarks

$$C_n = \frac{1}{2}(a_n - i b_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\underbrace{\left(\cos(nx) - i\sin(nx)\right)}_{= e^{-inx}} dx$$

$$C_{-n} = \frac{1}{2}(a_n + i b_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\underbrace{\left(\cos(nx) + i\sin(nx)\right)}_{= e^{inx}} dx$$

$$C_0 = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-i\cdot 0\cdot x}\,dx$$

Conclusion (★: $k \to \infty$)

Complex Fourier series for 2π-periodic function $f(x)$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}, \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx.$$

Note: Parseval's identity for this complex case is given by....

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} \underline{|C_n|^2}$$

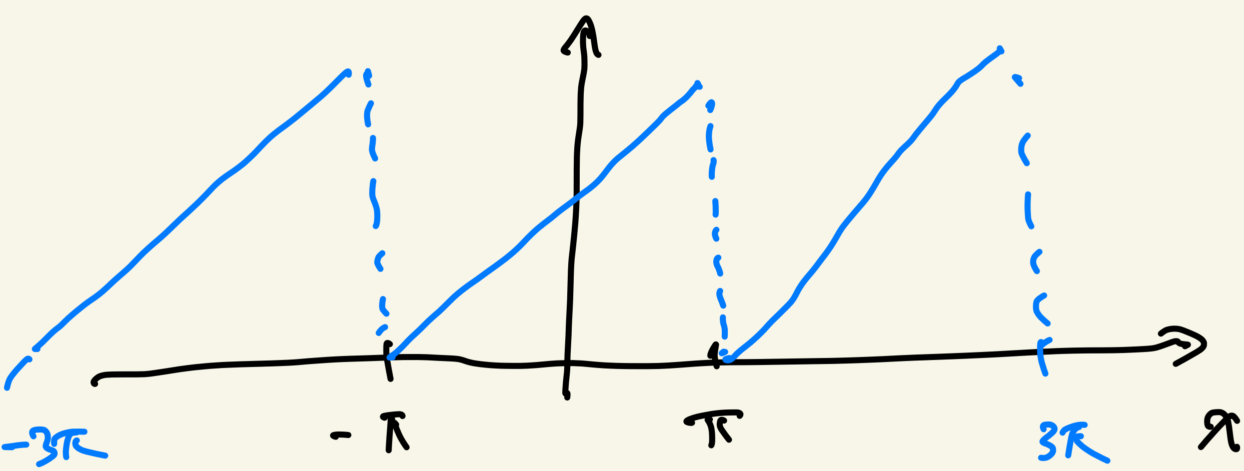
abs. value of complex numbers!!

$$|C_n|^2 + |C_{-n}|^2 = \frac{a_n^2 + b_n^2}{2}.$$

$$\left[\begin{array}{ll} C = x + iy \in \mathbb{C} & \\ x, y \in \mathbb{R} & |C| = \sqrt{x^2 + y^2} \end{array} \right]$$

<u>EX3</u>. $f(x) = x + \pi, \quad -\pi < x < \pi.$

$f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$.



What are C_n's?

<u>$n \neq 0$</u>.

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (x+\pi)\, e^{-inx}\, dx$$

int. by parts

$$\overset{\varphi}{=} \frac{1}{2\pi}\left[(x+\pi)\frac{1}{-in}e^{-inx}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{(-in)}e^{-inx}\, dx$$

$\frac{1}{i} = -i$

$$\overset{\varphi}{=} \frac{1}{2\pi}\frac{i}{n}\left(2\pi e^{-in\pi} - 0\right) - \frac{1}{2\pi}\left(\frac{1}{n}\right)^2\left(e^{-in\pi} - e^{in\pi}\right)$$

$$= \frac{i}{n}\Big(\cos(-n\pi) + \underbrace{i\sin(-n\pi)}_{=0}\Big) + \frac{1}{2\pi n^2}\Big(-2i\underbrace{\sin(n\pi)}_{=0}\Big)$$

$$= \frac{i}{n}\cos(n\pi) \quad \left(= \frac{i}{n}(-1)^n\right)$$

$\underline{n=0}$ $$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+\pi)\, dx = \pi$$

Thus, the complex Fourier series of f is

$$S_f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i(-1)^n}{n} e^{inx}$$

//

Remark. Why do we have " e^{-inx} "

in

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-in\pi}\, dx \quad ?$$

$$a_n = \frac{1}{\pi}\langle f, \cos(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$$

$$(b_n = \text{ — '' — } \sin(nx) \text{ — '' — })$$

For complex functions, L^2 inner-product is defined by

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)\cdot \overline{g(x)}\, dx.$$

Therefore,

$$c_n = \frac{1}{2\pi}\langle f, e^{inx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot \underline{\overline{e^{inx}}}\, dx$$

$$\overline{e^{inx}} = \overline{\cos(nx) + i\sin(nx)} = \cos(nx) - i\sin(nx) = e^{-inx}.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx.$$

Lecture 7: Fourier Series

Approximation of $f(x)$ by trigonometric polynomial

$$P_k(x) = A_0 + \sum_{n=1}^{k}(A_n \cos nx + B_n \sin nx).$$

Mean square (L^2) error minimal when $P_k = S_{f,k}$:

$$\int_{-\pi}^{\pi} |f(x) - S_{f,k}(x)|^2 dx \leq \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx$$

for all $P_k(x)$

where the k-th Fourier partial sum $S_{f,k}$ is

$$S_{f,k}(x) = a_0 + \sum_{n=1}^{k}(a_n \cos nx + b_n \sin nx).$$

Lecture 7: Fourier Series

$$\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right]$$

Bessel's inequality

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

Lecture 7: Fourier Series

Fourier series of $2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Complex Fourier series of $2L$-periodic $f(x)$:

$$S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx$$

Summary: Fourier series

1. Fourier series of 2π-periodic $f(x)$:

   $$S_f(x) \;= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

   $$S_{f,k}(x) = a_0 + \sum_{n=1}^{k}(a_n \cos nx + b_n \sin nx) \qquad k\text{-th}$$

   partial sum

2. Approximation of $f(x)$ by trigonometric polynomial

   $$P_k(x) = A_0 + \sum_{n=1}^{k}(A_n \cos nx + B_n \sin nx)$$

   Mean square (or L^2) error:

   $$\boxed{\|f - P_k\|^2 := \int_{-\pi}^{\pi} |f(x) - P_k(x)|^2 dx}$$

   $S_{f,k}(x)$ best approximation (least error):

   $$\boxed{\|f - S_{f,k}\|^2 \le \|f - P_k\|^2} \quad \text{for all } P_k(x)$$

   Obs:

   $$\|f - S_{f,k}\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{k}(a_n^2 + b_n^2)\right]$$

Summary: Fourier series

5. Fourier series of $2L$-periodic $f(x)$:

$$S_f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

6. Complex Fourier series of $2L$-periodic $f(x)$:

$$\boxed{S_f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}} \qquad \boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} \, dx}$$

OBS: $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$!!!