Mathematics 4K (TMA4120)

Parallel 2: MTTK

Lecture 24

Summary Lecture 23: Singularities

1. Singularity: Point z_0 where $f(z)$ is not analytic/defined $(\dots)$

2. Isolated singularity:

   z_0 is the only singularity in $|z - z_0| < R$ for some $R > 0$

   $\Downarrow$ **Laurent's theorem**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\neq 0}, \quad 0 < |z - z_0| < R$$

Principal ~~value~~ Part of z_0: $\boxed{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}$

z_0 order n pole: $b_n \neq 0$, $b_k = 0$, $k > n$

z_0 isolated essential singularity: $b_n \neq 0$ for ∞ many n

Summary Lecture 23: Residue integration

3. Residue integration:

$$\boxed{\text{Goal: } \oint_C f(z)dz = 2\pi i \sum \text{residues}}$$

4. Only one singularity:

   z_0 only singularity of $f(z)$ enclosed by C (simple, closed, counter cl.wise)

$$\Downarrow \quad \textbf{Laurent's theorem}$$

$$f(z) = a_0 + a_1(z - z_0) + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots, 0 < |z - z_0| < R$$

$$\Downarrow \quad \textbf{Term wise integration/Laurent's theorem}$$

$$\boxed{\oint_C f(z)dz = 2\pi i b_1}$$

5. Residue of $f(z)$ at z_0: $\boxed{\operatorname*{Res}_{z=z_0} f(z) = b_1}$

   b_1-coefficient in Laurent series that converges in $0 < |z - z_0| < R$!

Lecture 24: Residue integration

$\boxed{\text{Kreyszig: Sections 16.3, 16.4}}$

1. Residue integration
2. Formulas for residue of a pole
3. Residue theorem
4. Application: Computing real integrals
5. Examples

Exercises 12 and 13:
Next week - Wednesday 23:59!

Info about exercises, exam, meeting time:
See wikipage.

Lecture 24

Residue integral

Ex1. (using $\oint_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_0} f(z)$

where only z_0 is the singularity in C.)

$C: |z| = \frac{1}{2}$, counter clockwise

$$f(z) = \frac{1}{z^3} \cdot \frac{1}{(1-z)}.$$

i) singularity at $z=0$ and $z=1$.

C encloses only $z=0$.

ii) Laurent series centered at $z=0$

$$\frac{1}{z^3} \cdot \frac{1}{1-z} = \begin{cases} \frac{1}{z^3} \sum_{n=0}^{\infty} z^n = \underline{\sum_{n=0}^{\infty} z^{n-3}}, & \underline{0<|z|<1} \\ \frac{1}{z^3} \cdot \left(- \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \right) = \sum_{n=0}^{\infty} \frac{-1}{z^{n+4}}, & |z|>1 \end{cases}$$

$\Leftarrow$ C is inside!

iii) $\oint_C f(z)\,dz = 2\pi i \underbrace{\operatorname{Res}_{z=z_0} f(z)}_{b_1 = 1} = 2\pi i$ //

Remark: We need to choose the "right" Laurent series! (not the one for $|z|>1$)

Residue of the order 1 pole
(simple pole)

Laurent series around an order 1 pole z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)}, \quad b_1 \neq 0$$

$$\Downarrow \times (z-z_0)$$

$$(z-z_0) f(z) = (z-z_0) \sum_{n=0}^{\infty} a_n (z-z_0)^n + b_1$$

Let $z \to z_0$

$$\boxed{(1) \quad \operatorname{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0) f(z)}$$

Ex2.

$$f(z) = \frac{4}{1+z^2} = \frac{4}{(z-i)(z+i)}$$: order 1 pole at $z = \pm i$

$$\operatorname*{Res}_{z=i} f(z) \overset{(1)}{=} \lim_{z\to i} \left((z-i) f(z)\right) = \lim_{z\to i} \frac{4}{z+i} = -2i$$

$$\operatorname*{Res}_{z=-i} f(z) \overset{(1)}{=} \lim_{z\to -i} \left((z+i) f(z)\right) = \lim_{z\to -i} \frac{4}{z-i} = 2i$$ //

Assume $f(z) = \frac{P(z)}{q(z)}$, P and q are analytic. q has order 1 zero at z_0, and $P(z_0) \neq 0$.

Then.

$$q(z) = (z-z_0) q'(z_0) + \frac{1}{2}(z-z_0)^2 q''(z_0) + \cdots$$

(Taylor series)

Thus.

$$\operatorname*{Res}_{z=z_0} f(z) \overset{(1)}{=} \lim_{z\to z_0} (z-z_0) \frac{P(z)}{q(z)}$$

$$= \lim_{z \to z_0} \frac{(z-z_0)\, p(z)}{(z-z_0)\left(q'(z_0) + \frac{1}{2}(z-z_0)\, q''(z_0) + \cdots\cdots\right)}$$

$$= \frac{p(z_0)}{q'(z_0)}$$

Thus...

$$(2) \quad \boxed{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

Ex3.

$$f(z) = \frac{1}{1-e^z}$$

$q(z) = 1 - e^z$, has order 1 zero at $z=0$

$(q'(0) \neq 0)$, p, q, are analytic.

$$\Rightarrow \operatorname*{Res}_{z=0} f(z) = \frac{p(0)}{q'(0)} = \frac{1}{-e^0} = -1 \;\#$$

Residue for order n pole z_0

Laurent series around an order n pole z_0:

$$f(z) = \sum_{k=1}^{\infty} a_k (z-z_0)^k + \frac{b_1}{(z-z_0)} + \cdots\cdots + \frac{b_n}{(z-z_0)^n}$$

$$\Big\Downarrow \times (z-z_0)^n$$

$$(z-z_0)^n f(z) = \sum_{k=0}^{\infty} (z-z_0)^{n+k} a_k + b_1 (z-z_0)^{n-1} + \cdots + b_n$$

$\Big\Downarrow$ differentiate $(n-1)$ times and let $z \to z_0$

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\left((z-z_0)^n f(z) \right)^{(n-1)} \right]$$

derivative!

...... (3)

Ex3. $f(z) = \dfrac{\cos(z)}{z^4}$: has order 4 pole at 0.

$$\operatorname*{Res}_{z=0} (f(z)) = \lim_{z \to 0} \frac{1}{3!} \big(\underbrace{z^4 f(z)}_{\cos(z)} \big)'''$$

$$= \frac{1}{6} \lim_{z \to 0} (\sin(z)) = 0.$$

Residue integral

Many singular points can be inside C.....

Thm 1 (Residue theorem)

Assume that $f(z)$ is analytic inside and on a simple closed curve C, except for finitely many singular points, $z_1, z_2, \cdots, z_m$ inside C.

If C oriented counter clockwise.

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z) \tag{4}$$

Ex 5.

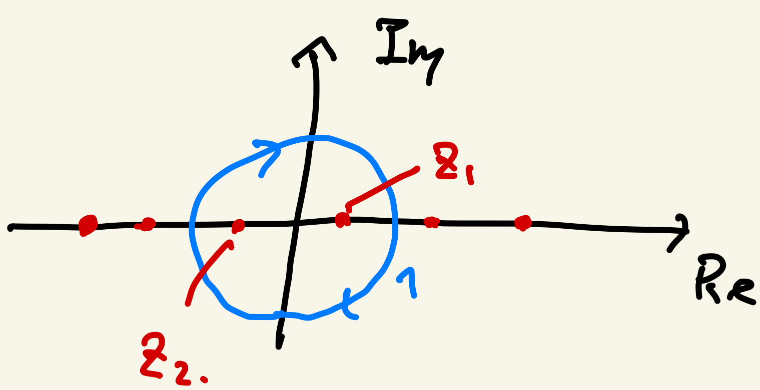
$$\oint_{|z|=1} \frac{1}{\cos(\pi z)}\,dz = ?$$

positively oriented.

(i) $\cos(\pi z)$ has order 1 zeros at $z = \frac{1}{2} + n$ $(n \in \mathbb{Z})$

$\frac{1}{\cos(\pi z)}$ has order 1 <u>poles</u> at —— || ——.

(ii) Inside $|z| = 1$, $z_1 = \frac{1}{2}$, $z_2 = -\frac{1}{2}$



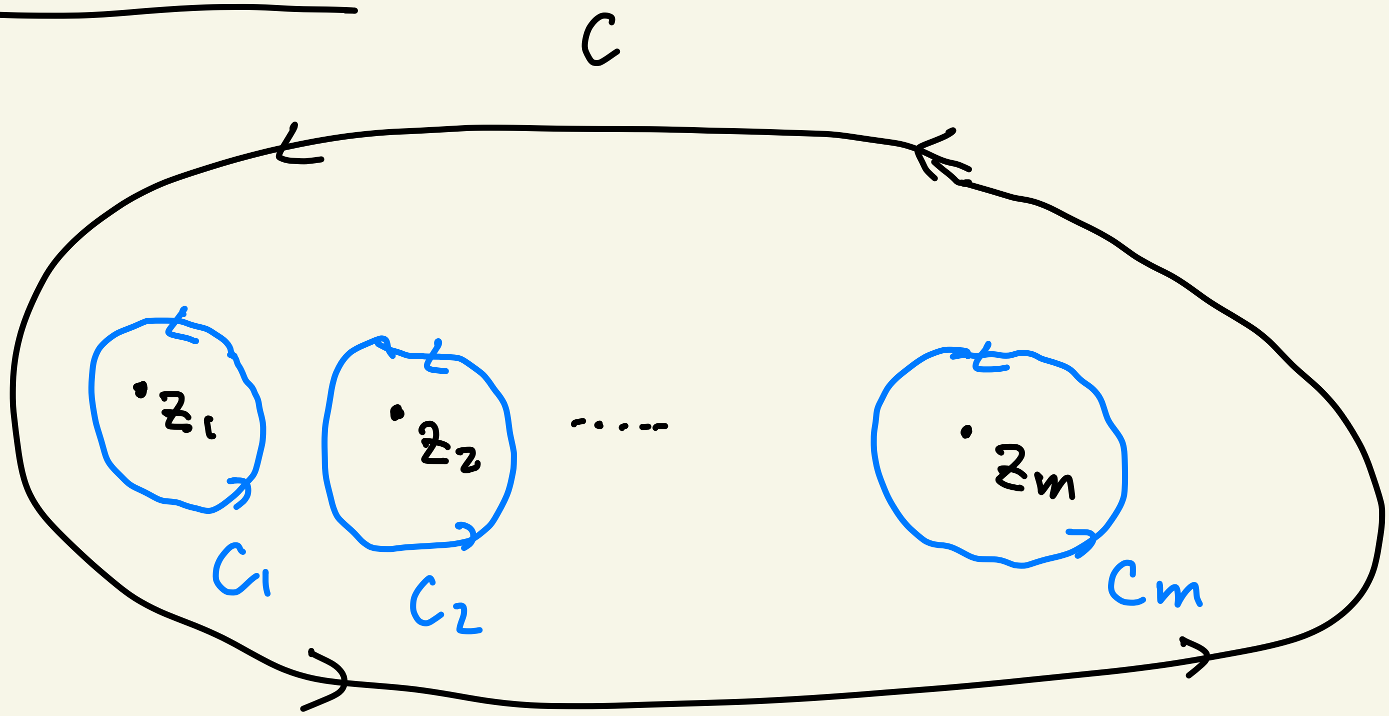
(iii)
$$\operatorname*{Res}_{z=\pm\frac{1}{2}}\left(\frac{1}{\cos(\pi z)}\right) \underset{\substack{4\\(2)}}{=} \left.\frac{1}{(\cos(\pi z))'}\right|_{z=\pm\frac{1}{2}}$$

$$= \left.\frac{1}{-\pi \sin(\pi z)}\right|_{z=\pm\frac{1}{2}} = \mp\frac{1}{\pi}.$$

(iv) Residue theorem: $\oint_{|z|=1} \frac{1}{\cos(\pi z)} dz = 2\pi i\left(-\frac{1}{\pi}+\frac{1}{\pi}\right)$

$= 0$ //

Proof of Thm 1)



Step 1) Enclose each z_j in a circle C_j inside C, where all C_j's are separated (not overlapping!)

Step 2) Use the Cauchy's integral theorem with m holes:

$$\oint_C f(z)\,dz = \sum_{j=1}^{m} \oint_{C_j} f(z)\,dz.$$

Step3). Laurent's theorem with the definition of residue...

$$\oint_{C_j} f(z)\,dz = 2\pi i\, b_1 = 2\pi i \operatorname*{Res}_{z=z_j} f(z)$$

for $j = 1, \dots, m$. ///

Ex6. [Exam 2002. Aug] $\boxed{VI a}$

$$\oint_{|z-i|=1} e^{\frac{z}{z-i}}\,dz = ?$$

(positively orient.)

Note: $f(z) = e^{\frac{z}{z-i}} = e^{\frac{i}{z-i}+1} = e^{\frac{i}{z-i}} \cdot e.$

i) Since $e^{\frac{1}{u}}$ has iso. ess. sing. at $u=0$.

$f(z)$ has iso. ess. sing at $z=i$.

ii) Singularity $z=i$ is enclosed in $|z-i|=1$.

iii) Laurent series:

$$f(z) \underset{u=\frac{i}{z-i}}{=} e\cdot e^{u} \underset{(|u|<\infty)}{=} e\sum_{n=0}^{\infty}\frac{u^n}{n!} \underset{u=\frac{i}{z-i}}{=} e\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{i^n}{(z-i)^n}$$

$$\left(\left|\frac{i}{z-i}\right|<\infty\right)$$

$$= e + \underbrace{e\cdot\frac{i}{z-i}}_{=\frac{b_1}{(z-i)}} + \frac{e\cdot i^2}{2}\cdot\frac{1}{(z-i)^2} + \cdots\cdots$$

$$(|z-i|>0)$$

Thus, $b_1 = ei$ $\left(= \operatorname{Res}_{z=i} f(z)\right)$

iv) Using the residue theorem

$$\oint_{|z-i|=1} f(z)\,dz = 2\pi i\,(ei) = -2\pi e$$

Remark: Iso. ess. sing. $\rightarrow$ We must find the right Laurent series to find b_1.

Ex 7. [Exam Dec. 2007]

i) Find all singularities and residues of

$$f(z) = \frac{(z+1)^2}{z(2z+1)^2}.$$

$$\rightarrow \begin{cases} f(z) \text{ has order } 1 \text{ pole at } z=0 \\ \text{—— 〃 ——} \ 2 \text{ pole at } z = -\frac{1}{2}. \end{cases}$$

$$\operatorname*{Res}_{z=0}\left(f(z)\right) \underset{(1)}{=} \lim_{z\to 0}\left(z\cdot f(z)\right) = \lim_{z\to 0} \frac{(z+1)^2}{(2z+1)^2} = 1.$$

$$\operatorname*{Res}_{z=-\frac{1}{2}}\left(f(z)\right) \underset{(3)}{=} \lim_{z\to -\frac{1}{2}} \left[\left[\left(z+\tfrac{1}{2}\right)^2 f(z)\right]'\right]$$

$$= \lim_{z\to -\frac{1}{2}} \left[\left(\frac{1}{4}\cdot\frac{(z+1)^2}{z}\right)'\right]$$

$$= \lim_{z\to -\frac{1}{2}} \left[\frac{1}{4}\cdot\left(1 - \frac{1}{z^2}\right)\right] = -\frac{3}{4} \quad //$$

ii) Calculate

$$I = \int_0^{2\pi} \frac{(e^{i\theta}+1)^2}{(2e^{i\theta}+1)^2}\, d\theta$$

By letting $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta = iz\, d\theta$.

Now $\theta \in [0, 2\pi)$ means we are integrating over the curve $z(\theta) = e^{i\theta}$: unit circle $|z|=1$ (counter clockwise)

$$I = \oint_{|z|=1} \frac{(z+1)^2}{(2z+1)^2} \frac{dz}{iz} = \frac{1}{i} \oint_{|z|=1} f(z)\, dz$$

$$\underset{\text{residue theorem!}}{=} \frac{1}{i} 2\pi i \left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=-\frac{1}{2}} f(z) \right)$$

$$= 2\pi \left(1 - \frac{3}{4}\right) = \frac{\pi}{2}$$

Calculating real integrals

We consider the integral of the form:

$$I = \int_0^{2\pi} F(\cos(\theta), \sin(\theta))\, d\theta$$

F: rational function.

Ex 8. [Exam. 2009. Dec]

$$F(\cos(\theta), \sin(\theta)) = \frac{1}{5+4\sin(\theta)}$$

Idea: Let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$

$C = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ unit circle, positively oriented.

$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z})$

$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z})$

and.

$$\boxed{I = \oint_C F\left(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})\right) \cdot \frac{dz}{iz}}$$

in our case...

$$I = \int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)} = \oint_{|z|=1} \frac{\frac{dz}{iz}}{5+4\frac{1}{2i}(z-\frac{1}{z})}$$

$$= \oint_{|z|=1} \underbrace{\frac{dz}{2z^2+5iz-2}}_{g(z)\,dz.}$$

Now, $2z^2+5iz-2=0$, $\Rightarrow z=-2i, -\frac{i}{2}$.

Inside $|z|=1$, $z=-\frac{i}{2}$ is the only singularity

Thus, $\operatorname{Res}_{z=-\frac{i}{2}} g(z) = \lim_{z\to -\frac{i}{2}} (z+\frac{i}{2})\,g(z) = \frac{1}{3i}$.

by using the residue thm..

$$I = 2\pi i \cdot \operatorname{Res}_{z=-\frac{i}{2}} g(z) = \frac{2}{3}\pi$$

//.

Lecture 24: Residues

$\operatorname*{Res}_{z=z_0} f(z) = b_1$ where

$$f(z) = \sum a_n(z-z_0)^n + \tfrac{b_1}{z-z_0} + \tfrac{b_2}{(z-z_0)^2} + \ldots, \quad 0 < |z-z_0| < R$$

z_0 order 1 pole:

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z) \quad \text{or} \quad \operatorname*{Res}_{z=z_0} \tfrac{p(z)}{q(z)} = \tfrac{p(z_0)}{q'(z_0)}$$

z_0 order n pole:

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z\to z_0} \Big((z-z_0)^n f(z)\Big)^{(n-1)}$$

z_0 isolated essential singularity:

Find b_1 from the Laurent series!

Lecture 24: The Residue theorem

Assume

(A1) C simple, closed curve oriented counterclockwise

(A2) $f(z)$ has singularities $z_1, \ldots, z_m$ enclosed by C

Then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z).$$

Summary Lecture 24: Residue integration

1. Residues: $\operatorname*{Res}_{z=z_0} f(z) = b_1$ where

$$f(z) = \sum a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots, \; 0 < |z-z_0| < R$$

   z_0 order 1 pole:

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0}(z-z_0)f(z)} \quad \text{or} \quad \boxed{\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

   z_0 order n pole:

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z\to z_0} \Big((z-z_0)^n f(z)\Big)^{(n-1)}}$$

   z_0 isolated essential singularity:

   Find b_1 from the Laurent series!

2. Residue theorem: Assume
   (A1) C simple, closed curve oriented counterclockwisely
   (A2) $f(z)$ has finite number of singularities $z_1, \ldots, z_m$ enclosed by C

   Then

$$\boxed{\oint_C f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z)}$$