Mathematics 4K (TMA4120)

Parallel 2: MTTK

Lecture 19

Summary Lecture 18: Complex line integrals

1. Complex line integral:

   $$\int_C [af(z) + bg(z)]dz = a\int_C f(z)dz + b\int_C g(z)dz$$

   $$\int_{-C} f(z)dz = -\int_C f(z)dz$$

   $$\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad \text{when} \quad C_1 \cap C_2 = \emptyset$$

   $$\boxed{\left|\int_C f(z)dz\right| \leq M \cdot L} \quad M = \max_{z \in C} |f(z)|, \quad L = \text{length of } C$$

2. Cauchy's integral theorem

   f analytic in simply connected domain D, $C \subset D$ simple, closed curv

   curve

   $$\Rightarrow \quad \oint_C f(z)dz = 0$$

3. Consequences:

   (a) $\int_C f(z)dz$ is independent of path in D

   (b) The indefinite integral of f exists in D, i.e. a function F s.t.

   $$F'(z) = f(z) \qquad \text{and} \qquad \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_2).$$

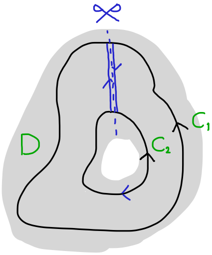
Summary Lecture 18: Domains with holes

4. Domains with holes:

   Cut to have a simlply connected domain...

   ... add segments along cut to have closed curve...

   Then use Cauchy:



Cauchy in the cut domain D^* and cancelations along cut:

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Summary Lecture 18: Example

C_1 any simple, closed curve surrounding $z = z_0$

C_1 : circle $|z - z_0|^2 = r^2$

Cauchy's integral theorem in domain with one hole, $m \in \mathbb{Z}$:

$$\oint_{C_1} (z - z_0)^m dz = \oint_{C_2 : z(t) = z_0 + re^{it}} (z - z_0)^m dz$$

$$\underset{Ex.1}{\overset{\text{last time}}{=}} \begin{cases} 2\pi i, & m = -1, \\ 0, & m \neq -1, \; m \in \mathbb{Z} \end{cases}$$

Lecture 19: Complex Analysis

Kreyszig: Sections 14.3, 14.4

1. Cauchy integral formula
2. Analytic functions infinitely differentiable
3. Properties of analytic functions:
   Cauchys inequality and Liouvilles theorem
4. Examples

Lecture 19

Cauchy's integral formula (important!)

Thm 1: (Cauchy's int. formula)

Assume:

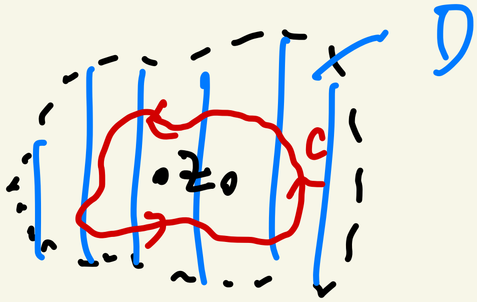
(A1) $f(z)$ is analytic in a simply connected domain D.

(A2) $z_0 \in D$, $C \subset D$ simple closed path, enclosing the point z_0, oriented to counter clockwise

["positively oriented"]

Then

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0) \qquad (1)$$



Remark) If the integration is taken

clockwise $\rightarrow$ $\displaystyle\oint_C \cdots\, dz = -2\pi i f(z_0)$

$\Updownarrow$

"negatively oriented"

Ex1. C: simple closed path enclosing $z = \frac{1}{3}$, counter clockwise

$$\oint_C \frac{z^3 \cos(z)}{3z-1}\, dz = \oint_C \frac{\overbrace{\left[\frac{1}{3} z^3 \cos(z)\right]}^{f(z)}}{z - \frac{1}{3}}\, dz.$$

$$\underset{(1)}{\overset{\Updownarrow}{=}} 2\pi i \left[\frac{1}{3} z^3 \cos(z)\right]_{z=\frac{1}{3}} = \frac{2\pi i}{81} \cos\left(\frac{1}{3}\right)$$

$$\approx 0.095\, i$$

//

Proof of Thm1).

$$\oint_C \frac{f(z)}{z-z_0}\, dz = \oint_C \frac{f(z) - f(z_0) + f(z_0)}{z - z_0}\, dz.$$

$$= \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz + \underbrace{\oint_C \frac{f(z_0)}{z - z_0}\, dz}_{\shortparallel}.$$

$$\parallel$$

$$2\pi i \cdot f(z_0)$$

Now we want to show that the first term $= 0$.

$\dfrac{f(z)-f(z_0)}{z-z_0}$ is analytic except for $z = z_0$ in D.

Let a circle $C_\rho : |z - z_0| = \rho$, with small $\rho > 0$ such that C_ρ is enclosed in C.

Then the Cauchy's integral theorem tells us.

$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = \oint_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

Using the ML-inequality:

$$\left| \oint_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \le \underbrace{\left[\max_{z\in C_\rho} \frac{|f(z)-f(z_0)|}{\underbrace{|z-z_0|}_{\to \cancel{\rho}}} \right]}_{M} \underbrace{2\pi\cancel{\rho}}_{L}$$

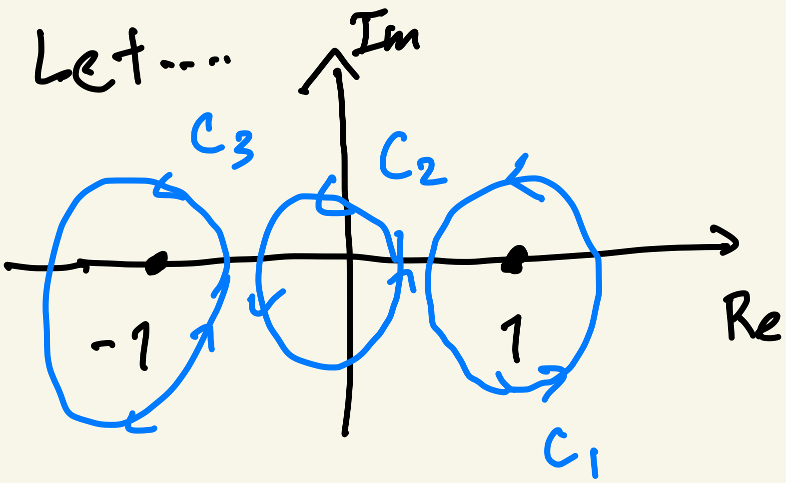
$$= 2\pi \max_{z\in C_\rho} |f(z)-f(z_0)| \xrightarrow[\rho\to 0+]{} 0$$

since f is continuous.

EX2.

Let $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z-1)(z+1)}$ is analytic except for $z = \pm 1$.

Let ....



$$\oint_{C_1} g(z)\,dz = \oint_{C_1} \frac{1}{z-1} \cdot \frac{z^2+1}{z+1}\,dz.$$

$$\overset{\text{Thm 1}}{=} 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i$$

$$\oint_{C_2} g(z)\,dz \underset{\uparrow}{=} 0$$

Cauchy's int. theorem.

$$\oint_{C_3} g(z)\,dz = \oint_{C_3} \frac{1}{z+1} \cdot \frac{z^2+1}{z-1}\,dz$$

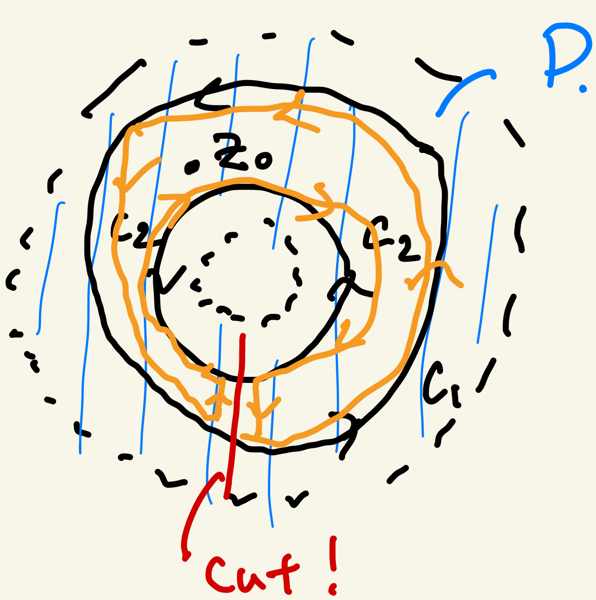
$$\underset{\text{Thm 1.}}{=} 2\pi i \cdot \left[\frac{z^2+1}{z-1} \right]_{z=-1} = -2\pi i \quad \#.$$

Ex 3 . For simple closed path C enclosing z_1 and z_2, positively oriented, and f is analytic on and inside C, then.

$$\oint_C \frac{f(z)}{(z-z_1)(z-z_2)}\,dz \underset{\text{partial frac. decomp.}}{=} \oint_C f(z)\left(\frac{A_1}{z-z_1} + \frac{A_2}{z-z_2}\right)dz$$

$$\underset{\text{Thm 1.}}{=} 2\pi i\left(A_1 f(z_1) + A_2 f(z_2)\right)$$

Domain with a hole



Cauchy's int. formula

$$\oint \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\Leftrightarrow\ 2\pi i\, f(z_0) = \oint_{C_1} \frac{f(z)}{z-z_0}\,dz - \oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

Derivatives of analytic functions

Thm 2.

$f(z)$ is analytic in a simply connected domain D.

$$\Downarrow$$

$f^{(n)}$, $n \in \mathbb{N}$ exists and are analytic in D.

To prove this, we need the following theorem:

Thm 3: Assume...

(A1)': f continuous in a simply connected domain D.

(A3): For every point $z_0 \in D$, and every simple closed path $C \subset D$ which is positively oriented and enclosing z_0; $f^{(n-1)}(z_0)$ exists and

$$f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^n}\, dz$$

Then $f^{(n)}(z_0)$ exists in every $z_0 \in D$, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz. \tag{2}$$

Proof of Thm 3).

When $\underline{n=1}$. we want to show

$$\lim_{\Delta z \to 0} \left(\text{following}\right) = 0.$$

$$\left| \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz \right|$$

(A3) (A3)

$$= \left| \frac{1}{2\pi i} \oint_C \frac{1}{\Delta z} \frac{f(z)}{z-(z_0+\Delta z)} - \frac{1}{\Delta z} \frac{f(z)}{z-z_0} - \frac{f(z)}{(z-z_0)^2}\, dz \right|$$

Sum up!

$$\underset{\text{(Check)}}{=} \left| \frac{1}{2\pi i} \oint_C \frac{f(z)\,\Delta z}{(z-z_0-\Delta z)\cdot(z-z_0)^2}\, dz \right|$$

$$\underset{\text{ML-inequality}}{\leq} \frac{1}{2\pi} \underbrace{\max_{z\in C} \frac{|f(z)|\cdot|\Delta z|}{|z-z_0-\Delta z|\cdot|z-z_0|^2}}_{M} \cdot L$$

[Now, let d be the smallest distance between z_0 and curve C.
d is a constant, so we can choose $|\Delta z| \leq \frac{d}{2}$]

$$\left(\Rightarrow |z-z_0|^2 \geq d^2,\ |z-z_0-\Delta z| \geq \frac{d}{2}\right)$$

$$\leq \frac{1}{2\pi}\cdot\left[\max_{z\in C}|f(z)|\right]\frac{|\Delta z|}{\frac{d}{2}\cdot d^2}\cdot L \xrightarrow[\Delta z\to 0]{} 0.$$

So we proved for $n=1$.

For $n\geq 2$, repeat the same argument for $f^{(n-1)}$ ...

$\blacksquare$

Proof of Thm 2)

.

f is analytic $\overset{\text{(Thm 1)}}{\Longrightarrow}$ (A1)$'$ and (A3) hold.
for $n=1$.

$\overset{\text{(Thm 3)}}{\Longrightarrow}$ f' exists and (2) hold
for $n=1$.

$\overset{\text{(Thm 1)}}{\Longrightarrow}$ (A1)$'$ and (A3) hold for $n=2$.

$\overset{\text{(Thm 3)}}{\Longrightarrow}$ f'' exists and (2) hold (2)
for $n=2$

$\Longrightarrow \cdots\cdots$

(by induction) $\blacksquare$

Remarks

i) f is analytic $\Rightarrow$ $\begin{cases} f \text{ is } \infty \text{ differentiable.} \\ f^{(n)} \text{ satisfies (2)} \\ \qquad \text{for any } n \in \mathbb{N}. \end{cases}$

ii) $$\frac{d^n}{dz_0^n} f(z_0) \overset{\underline{\text{Thm 1}}}{\underset{\downarrow}{=}} \frac{1}{2\pi i} \frac{d^n}{dz_0^n} \oint_C \frac{f(z)}{(z-z_0)} dz$$

$$\overset{\text{"}}{=} \frac{1}{2\pi i} \oint_C \frac{d^n}{dz_0^n} \frac{f(z)}{(z-z_0)} dz$$

(Ikke pensum)

$$= \frac{1}{2\pi i} \oint_C \frac{n!\, f(z)}{(z-z_0)^{n+1}} dz .$$

$(\rightarrow (2))$

Ex 4. Let a closed path C enclosing $z=2$, and C is in $Re(z) > 0$ and positively oriented.

then $$\oint_C \frac{Ln(z)}{(z-2)^3} dz \overset{\text{Thm 3}}{=} \frac{2\pi i}{2!} \left[Ln(z)'' \right]_{z=2}$$

$$= \pi i \cdot \left[-\frac{1}{z^2} \right]_{z=2} = \frac{-\pi i}{4}$$

Properties of analytic functions

1) Cauchy's inequality.

If f is analytic in a simply connected domain D, which includes $C_r : |z-z_0| = r$, then

$$|f^{(n)}(z_0)| \overset{\text{Thm3}}{=} \frac{n!}{2\pi}\left|\oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

$$\underset{\text{ML-ineq.}}{\leq} \underbrace{\frac{n!}{2\pi} \cdot \frac{1}{r^{n+1}} \cdot \max_{z \in C_r} |f(z)|}_{M} \cdot \underbrace{2\pi r}_{L}$$

Thus.

$$\boxed{|f^{(n)}(z_0)| \leq n! \cdot \frac{1}{r^n} \cdot \max_{z \in C_r} |f(z)|}$$

2) Liouville's theorem

If f is an entire function, and
(analytic over $\mathbb{C}$)
bounded $|f(z)| \leq K$ for any $z \in \mathbb{C}$, then.

f is a constant.

proof) $|f'(z_0)| \underset{\text{Cauchy's ineq.}}{\leq} \frac{1}{r} \cdot K \xrightarrow[r \to \infty]{} 0$

$\Rightarrow f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$.

$\Rightarrow f$ is constant. ▨

(3) Morera's theorem [converse of Cauchy's int. theorem]

If f is continuous in a simply connected domain D.
and if $\oint_C f(z)\,dz = 0$
for every closed path C in D.
then $f(z)$ is analytic in D.

(proof omitted.)

(4) Gauss mean value theorem

If f is analytic in a simply connected domain D, which includes $C_r : |z - z_0| = r$.
then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$

Mean value of f over the circle $t \in [0, 2\pi]$

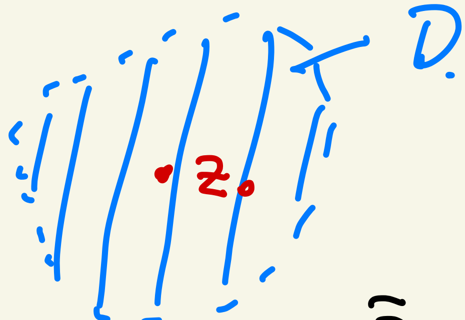
proof) $f(z_0) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0+e^{it})}{\cancel{r}e^{it}} ie^{it}\cancel{r} \, dt$

5) Maximum modulus principle

If f is analytic in a domain D,
and if there exists $z_0 \in D$ such that

$$\max_{z \in \overline{D}} |f(z)|$$

$\overline{D}$: D + boundary (closed).



$\overline{D}$: including boundary

then, f is constant in D.

(proof omitted)

Lecture 19: Cauchy's integral formula

f is analytic in simply connected domain D

$z_0 \in D$, $C \subset D$ simple closed curve,
positively oriented, enclosing z_0

Then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Lecture 19: Analytic => infinitly differentiable

f analytic in $D \quad \Rightarrow \quad f$ ∞-differentiable in D, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Lecture 19: Properties of analytic functions

Cauchy's inequality:

f analytic in $|z - z_0| \le r \quad \Longrightarrow \quad |f^{(n)}(z_0)| \le \dfrac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|$

Gauss mean value theorem:

f analytic in $|z - z_0| \le r \quad \Longrightarrow \quad f(z_0) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(z_0 + re^{it})\, dt$

Liouville's theorem:

f analytic, bounded in $\mathbb{C} \quad \Longrightarrow \quad f$ is constant

Maximum (modulus) principle:

f analytic in domain D and $|f(z)|$ attains its max in D
$\Longrightarrow \quad f$ is constant in D

Morera's theorem:

f continuous in D and $\oint_C f(z)dz = 0$ for all simple, closed $C \subset D$
$\Longrightarrow \quad f$ analytic in D

Summary Lecture 19

1. Cauchy's integral formula: $\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz}$ if

   (A1) f is analytic in simply connected domain D

   (A2) $z_0 \in D$, $C \subset D$ simple closed curve, positively oriented, enclosing z_0.

2. Infinitly differentiable:

   f analytic in D $\Rightarrow$ f infinitely differentiable in D, and

   $$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz}$$

3. Properties of analytic functions:

   Cauchy's inequality: $\boxed{|f^{(n)}(z_0)| \le \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|}$ if f analytic

   Liouville's theorem: f analytic, bounded in $\mathbb{C}$ $\Rightarrow$ f is constant

   Morera's theorem: f continuous in D and $\oint_C f(z)dz = 0$ for all simple, closed $C \subset D$ $\Rightarrow$ f analytic in D