Mathematics 4K (TMA4120)

Parallel 2: MTTK

Lecture 17

Summary: Complex Analysis

1. Exponential function:

$$\boxed{e^z = e^{x+iy} = e^x(\cos y + i \sin y)} \qquad 2\pi i\text{-periodic}$$

$$|e^z| = e^x, \qquad |e^{iy}| = 1, \qquad \arg e^z = y + 2\pi n,\ n \in \mathbb{Z}$$

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

$$(e^z)' = e^z \quad \text{(analytic in } \mathbb{C}\text{)}, \qquad e^z \neq 0 \text{ in } \mathbb{C} \quad \text{(conformal in } \mathbb{C}\text{)}$$

2. Trigonometric and hyperbolic functions:

$$\boxed{\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz})} \quad \boxed{\sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz})} \qquad 2\pi \text{ periodic}$$

$$\boxed{\cosh z = \tfrac{1}{2}(e^{z} + e^{-z})} \quad \boxed{\sinh z = \tfrac{1}{2}(e^{z} - e^{-z})} \qquad 2\pi i \text{ periodic}$$

$$\boxed{\tan z = \frac{\sin z}{\cos z}} \qquad \ldots$$

$$\cos^2 z + \sin^2 z = 1 \qquad \text{and} \qquad \cosh^2 z - \sinh^2 z = 1$$

$$(\cos z)' = -\sin z, \ldots \qquad \text{derivation as for real functions}$$

3. Logarithm:

$$\boxed{\ln z = \ln|z| + i(\operatorname{Arg} z + 2\pi n), \quad n \in \mathbb{Z}}$$

$$\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z \qquad \text{(principal value)}$$

Lecture 17: Complex Analysis

$\boxed{\text{Kreyszig: Sections 13.7, 14.1}}$

1. Logarithm
2. Complex Line integral
3. Examples

Lecture 17.

Complex logarithm.

Def of $\ln(z)$....

$$w = \ln(z) \overset{def}{\iff} e^{w} = z$$

then.

$$\ln(z) = \ln(|z|) + i(\mathrm{Arg}(z) + 2n\pi), \quad n \in \mathbb{Z}.$$

Definition: principal value of log.

$$\mathrm{Ln}(z) = \underbrace{\ln(|z|)}_{\text{real log. func.}} + i\,\underbrace{\mathrm{Arg}(z)}_{\text{principal value of argument } (-\pi, \pi]}$$

Remark).

i) Ln is uniquely defined (a function)

ii) Arg(z) and Ln(z) are discontinuous

over the negative real axis (including the origin).

Ex1.

$\ln(1) = 2n\pi i, \quad n \in \mathbb{Z}$

$\mathrm{Ln}(1) = 0, \quad (-\pi < \mathrm{Arg}(z) \leq \pi)$

$\ln(-1) = (2n+1)\pi i, \quad n \in \mathbb{Z}.$

$\mathrm{Ln}(-1) = \pi i$

$\ln(3-4i) = \ln(5) + i(-0.927 + 2n\pi), \quad n \in \mathbb{Z}.$

$\mathrm{Ln}(3-4i) = \ln(5) - 0.927\, i.$

Remark.

$$\begin{cases} \ln(z_1 z_2) = \ln(z_1) + \ln(z_2) \\ \ln\left(\frac{z_1}{z_2}\right) = \ln(z_1) - \ln(z_2) \end{cases}$$

$\hookrightarrow$ these should be understood as a set: both hand sides contain ∞ many values.

(→ these don't hold for the principal value Ln !!)

Ex2. Let $z_1 = z_2 = e^{i\pi} = -1$.

$\ln(z_1 z_2) = \ln(1) = 2n\pi i, \quad n \in \mathbb{Z}$.

$\ln(z_1) + \ln(z_2) = (2n_1+1)\pi i + (2n_2+1)\pi i$

$(n_1, n_2 \in \mathbb{Z})$

$= 2\pi i + 2(n_1+n_2)\pi i$

$= 2\pi i + 2n\pi i, \quad n \in \mathbb{Z}$.

but ....

$Ln(z_1 z_2) = 0 \neq Ln(z_1) + Ln(z_2) = \pi i + \pi i$ //

$$\boxed{\text{Theorem: Except for } z=0 \text{ and negative real axis, } Ln(z) \text{ is analytic and } (Ln(z))' = \frac{1}{z}}$$

proof) Ln is continuous and defined for the region of consideration ($\overline{}$)

$$w = \mathrm{Ln}(z) = \ln(|z|) + i\,\mathrm{Arg}(z)$$

$$= \underbrace{\frac{1}{2}\ln(x^2+y^2)}_{u} + i\,\overbrace{\underbrace{\arctan\left(\frac{y}{x}\right)}_{\varphi}}^{v}$$

φ

$z = x+iy$

$\left(\text{or, } \arctan\left(\frac{y}{x}\right)+\pi \text{ for } \mathrm{Im}(z)<0 \right)$

Thus.

$$\left.\begin{aligned} u_x &= \frac{x}{x^2+y^2} = v_y \\ u_y &= \frac{y}{x^2+y^2} = -v_x \end{aligned}\right\} \text{(check!)}$$

$\Rightarrow$ Ln is analytic, since the Cauchy-Riemann equation holds.

And $(\mathrm{Ln}(z))' = u_x + i v_x = \dfrac{x-iy}{x^2+y^2} = \dfrac{\bar{z}}{z \cdot \bar{z}}$

$= \dfrac{1}{z}$. ∎

Ln as a mapping

(a) $\{z : -\pi < \mathrm{Im}(z) \le \pi\} \xrightarrow{e^z} \mathbb{C} \setminus \{0\}$

$\Downarrow$

$\mathbb{C} \setminus \{0\} \xrightarrow{w = \mathrm{Ln}(z)} \{z : -\pi < \mathrm{Im}(w) \le \pi\}$

($\mathrm{Ln}(z)$ is the inverse of e^z)

(b) For $n \in \mathbb{Z}$: $\mathrm{Ln}(z) + 2n\pi i = w$

$\mathbb{C} \setminus \{0\} \longrightarrow \{z : -\pi + 2n\pi < \mathrm{Im}(w) \le \pi + 2n\pi\}$

Because $\ln(z) = \mathrm{Ln}(z) + 2n\pi i$, $n \in \mathbb{Z}$.

we have $\mathbb{C} \setminus \{0\} \xrightarrow{\ln(z) = w} \mathbb{C}$.

General power

Definition.

$$z^c = e^{c\ln(z)} \quad (c \in \mathbb{C}),\ (z \neq 0)$$

$$a^z = e^{z\ln(a)} \quad (\underset{\neq 0}{a} \in \mathbb{C})$$

Remark.

$c \in \mathbb{Z} \Rightarrow z^c$ have one value.

$c = \frac{1}{n},\ n \in \mathbb{Z},\ \Rightarrow z^c$ have n different values

(recall "n-th root")

Ex 3.

$$i^i = e^{i\ln(i)} = e^{i\left(i\frac{\pi}{2} + 2n\pi i\right)}$$

$$= e^{-\frac{\pi}{2} - 2n\pi}. \quad (n \in \mathbb{Z})$$

(Infinitely many Values!) #

Caution:

What went wrong in the following?

$$-1 = e^{i\pi} = e^{2\pi i \cdot \frac{1}{2}} = (1)^{\frac{1}{2}} = 1$$

ok. ok not ok. not ok.

$$\left(a^{h} = e^{h \underline{\ln(a)}} \right)$$

infinitely many values!

$$(1)^{\frac{1}{2}} = e^{\frac{1}{2}\ln(1)} = e^{n\pi i} = \begin{cases} 1 \\ -1 \end{cases}$$

Be careful about a^{h}, etc. as a complex number!

curve C on $\mathbb{C}$

Consider a parametrized curve C:

$z(t) = x(t) + iy(t)$, $t \in [a, b]$

x, y are continuous over t.

Definition

(i) Orientation of C: direction for increasing t.

(ii) C is smooth if

(a) $\dot{z}(t) = \dfrac{dz(t)}{dt} = \dot{x}(t) + i\dot{y}(t)$ exists and continuous.

(b) $\dot{z}(t) \neq 0$, $t \in [a, b]$.

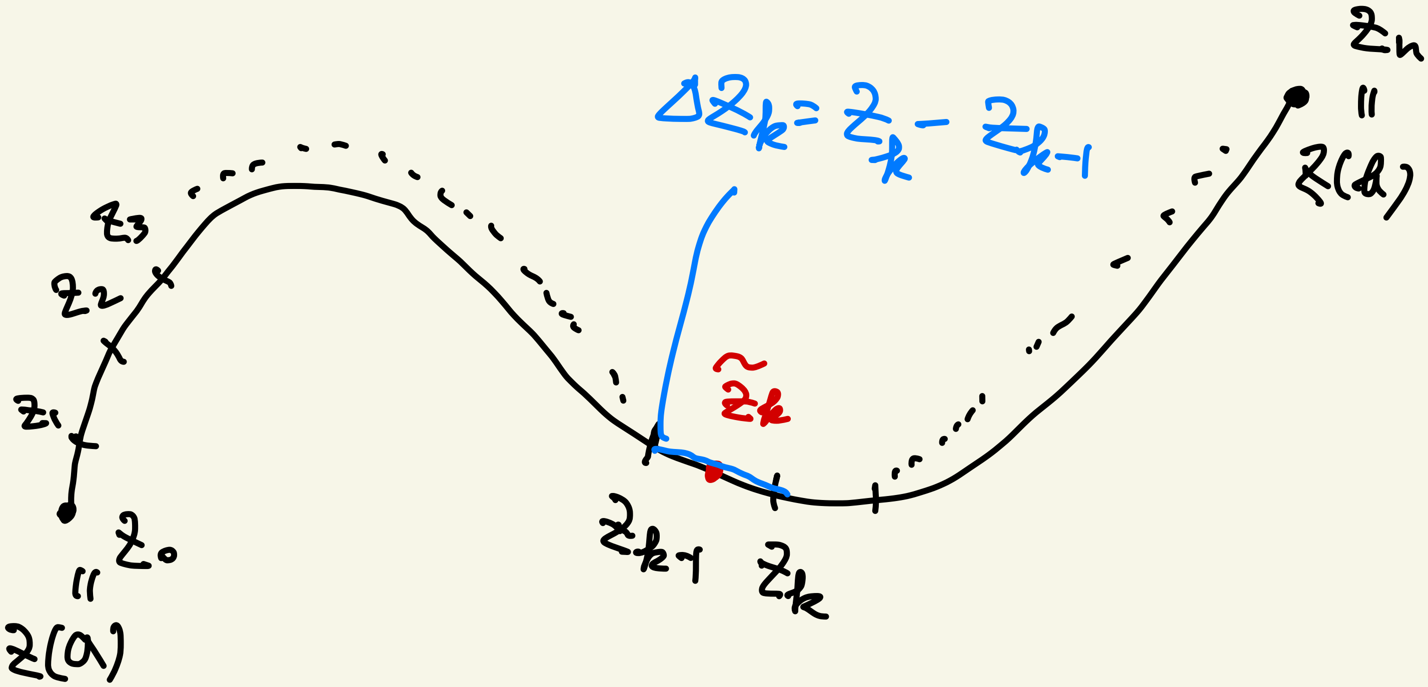
if C is smooth,

$\Rightarrow$ tangent $T(t) = \dfrac{\dot{z}(t)}{|\dot{z}(t)|}$ exists and continuous in $[a, b]$.

Complex line integral $\int_C f(z)\,dz$

("like real line integral ......")

Consider curve C:



Partition of C: $P_n = \{z_0, z_1, \ldots, z_n\}$

Selection: $\tilde{P}_n = \{\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n : \tilde{z}_k \in C$ and between z_{k-1} and z_k for $k = 1, \ldots, n\}$

Riemann Sum: $S_n = \sum_{k=1}^{n} f(\tilde{z}_k)\Delta z_k$

$(\Delta z_k = z_k - z_{k-1})$

Now assume: P_n are such points

$$\max_{k=1,\cdots,n} |\Delta z_k| \longrightarrow 0, \text{ as } n \to \infty$$

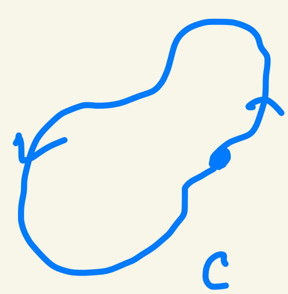
Definition: Line integral of f along C:

$$\int_C f(z)\,dz = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(\tilde{z}_k)\Delta z_k$$

[recall: real integral $\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy$]

Notation: when "initial point" = "terminal point"

we denote by $\oint_C f(z)\,dz$.



Theorem 1. Assume

(A1) C is piecewise smooth, oriented, finite length.

(A2) f is continuous on the curve C.

Then, there exists $\int_C f(z)dz$ independent of the choice of P_n and $\tilde{P}_n$

Proof) $S_n = \sum_{k=1}^{n} f(\tilde{z}_k)\Delta z_k$

$\overset{\Phi}{=} \sum_{k=1}^{n} (\underbrace{u(\tilde{x}_k,\tilde{y}_k)}_{u_k} + i \underbrace{v(\tilde{x}_k,\tilde{y}_k)}_{v_k})(\Delta x_k + i\Delta y_k)$

$\tilde{z}_k = \tilde{x}_k + i\tilde{y}_k$

$f = u + iv$

$\Delta z_k = \Delta x_k + i\Delta y_k$

$$= \underbrace{\sum_{k=1}^{n} u_k \Delta x_k - \sum_{k=1}^{n} v_k \Delta y_k + i\sum_{k=1}^{n} u_k \Delta y_k + \sum_{k=1}^{n} i v_k \Delta x_k} \quad -(1)$$

These sums are all real. Since f is continuous, u and v are also continuous on C.

Note that $|\Delta x_k|, |\Delta y_k| \leq |\Delta z_k|$

$\left(|\Delta z_k| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}\right)$

$$\Rightarrow \begin{cases} \max\limits_{k=1,\dots,n} |\Delta x_k| \xrightarrow[n\to\infty]{} 0 \\ \max\limits_{k=1,\dots,n} |\Delta y_k| \xrightarrow[n\to\infty]{} 0 \end{cases}$$

Therefore, the sum converges to a real line integral:

$$S_n = (1) \xrightarrow[n\to\infty]{} \int_C u\,dx - \int_C v\,dy + i\int_C u\,dy + i\int_C v\,dx.$$

$f = u + iv$
$dz = dx + idy$

$$= \int_C f(z)\,dz.$$

(not depending on the choice of P_n and $\hat{P}_n$)

Remark.

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$$

Theorem 2. Assume

(A1)' C: given by $z(t)$, $t \in [a,b]$, and piecewise smooth curve.

(A2) f: continuous on C.

Then $$\int_C f(z)\,dz = \int_a^b f(z(t))\cdot \dot{z}(t)\,dt.$$

"Proof"

t_k: $\hat{z}_k = z(t_k)$

$$\underbrace{\sum_{k=1}^{n} f(\hat{z}_k)\cdot \Delta z_k}_{\substack{\downarrow\, n\to\infty \\ \int_C f(z)\,dz}} = \sum_{k=1}^{n} f(z(t_k)) \underbrace{\int_{t_{k-1}}^{t_k} \dot{z}(s)\,ds}_{\substack{\downarrow\, n\to\infty \\ \dot{z}(t_k)\,\Delta t_k}}$$

$$\Big\downarrow n\to\infty$$

Riemann Sum

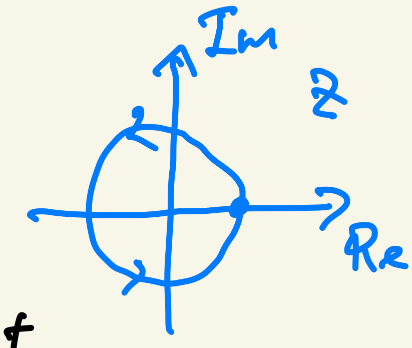
$$\int_a^b f(z(t))\cdot \dot{z}(t)\,dt.$$

$\boxtimes$

Ex4.

Let $C: z(t)=e^{it},\quad t\in[0,2\pi]$

$f(z)=\frac{1}{z}.$



$$\oint_C \frac{1}{z}\,dz \underset{\text{Thm2.}}{=} \int_0^{2\pi}\frac{1}{z(t)}\cdot \dot{z}(t)\,dt.$$

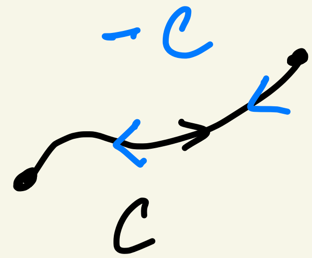
$$=\int_0^{2\pi}\cancel{e^{-it}}\cdot i\,\cancel{e^{it}}\,dt = 2\pi i$$ //

Properties [from the def.]

(a) $\displaystyle\int_C [a f(z)+b\,g(z)]\,dz = a\int_C f(z)\,dz + b\int_C g(z)\,dz.$

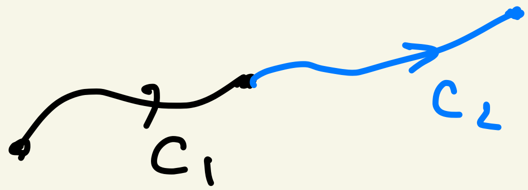
(linearity)

(b) $$\int_{\underline{-C}} f(z)\,dz = -\int_{C} f(z)\,dz.$$



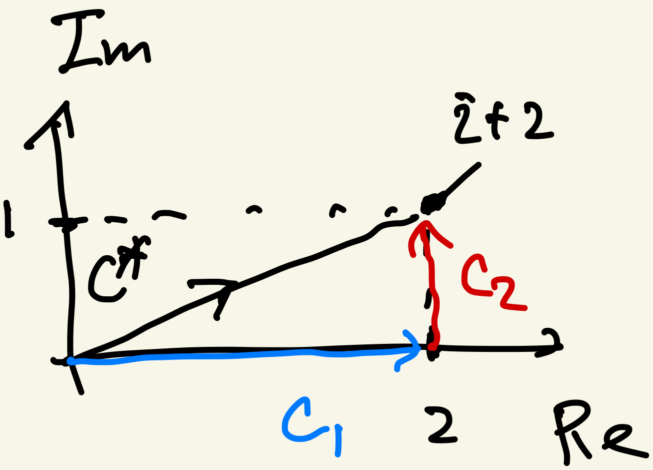
(c) $$\int_{C_1 \cup C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

$\cup$ union



Ex 5.

Let $f(z) = \mathrm{Re}(z)$.



$C^*: z(t) = t(2+i),\quad t \in [0, 1]$

$$\int_{C^*} f(z)\,dz = \int_0^1 f(z(t)) \cdot \dot{z}(t)\,dt$$

$$= \int_0^1 2t \cdot (2+i)\,dt = \underline{\underline{2+i}}$$

$$\int_{C_1 \cup C_2} f(z)\,dz \,?$$

$$C_1 : z_1(t) = t,\ t \in [0,2]$$

$$C_2 : z_2(t) = 2 + it,\ t \in [0,1]$$

Then,

$$\int_{C_1 \cup C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

$$\underset{\text{Thm. 2}}{=} \int_0^2 f(z_1(t))\,\dot{z}_1(t)\,dt + \int_0^1 f(z_2(t))\,\dot{z}_2(t)\,dt$$

$$= \int_0^2 t \cdot 1\,dt + \int_0^1 2 \cdot i\,dt = 2 + 2i.$$

Remark. Generally, $\int_C f(z)\,dz$ depends on the whole curve C, not only the endpoints

Lecture 17: Logarithm and general powers

$$\ln x = \ln|z| + i(\operatorname{Arg} z + 2\pi n), \qquad n \in \mathbb{Z}$$

$$\operatorname{Ln} x = \ln|z| + i\operatorname{Arg} z \quad \text{(principal value)}$$

$\boxed{\operatorname{Ln} z \text{ analytic}}$ except at $z = 0$ and negativ real axis

$$\boxed{(\operatorname{Ln} z)' = \frac{1}{z}}$$

$$z^c \overset{\text{DEF}}{=} e^{c \ln z}$$

Lecture 17: Complex line integral

Defined via Riemann-sums:

$$\int_C f(z)dz = \lim_{n\to\infty, |P_n|\to 0} \sum_{k=1}^{n} f(\tilde{z}_k)\Delta z_k$$

Exists and uniquely defined when:

(A1) C piecewise smooth, oriented, finite length

(A2) f is continuous on C

Along parametric curve $C : z(t), t \in [a, b]$:

$$\boxed{\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt}$$

Recall: - Parametric curve $C : z(t), t \in [a, b]$ smooth if $\dot{z}(t)$ exists, cont., $\neq 0$

- then unit tangent $T(t) = \frac{\dot{z}(t)}{|\dot{z}(t)|}$ exists and is continuous

Lecture 17: Complex line integral

$$\int_C [af(z) + bg(z)]dz = a\int_C f(z)dz + b\int_C g(z)dz$$

$$\int_{-C} f(z)dz = -\int_C f(z)dz$$

$$\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz, \quad C_1 \cap C_2 = \emptyset$$

Summary: Complex Analysis

1. **Logarithm:**

   $$\boxed{\ln x = \ln|z| + i(\operatorname{Arg} z + 2\pi n), \qquad n \in \mathbb{Z}}$$

   $\operatorname{Ln} x = \ln|z| + i\operatorname{Arg} z$ (principal value)

   $\boxed{(\operatorname{Ln} z)' = \frac{1}{z}}$ $\boxed{\operatorname{Ln} z \text{ analytic}}$ except at $z = 0$ and negativ real axis

   $z^c \overset{\text{DEF}}{=} e^{c \ln z}$

2. **Complex line integral:**

   Defined via Riemann-sums, exists and uniquely defined when:

   (A1) C piecewise smooth, oriented curve with finite length

   (A2) f is continuous on C

   $$\boxed{\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt} \quad \text{when } C : z(t), t \in [a, b]$$

   $$\int_C [af(z) + bg(z)]dz = a\int_C f(z)dz + b\int_C g(z)dz$$

   $$\int_{-C} f(z)dz = -\int_C f(z)dz$$

   $$\int_{C_1 \cup C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad \text{when} \quad C_1 \cap C_2 = \emptyset$$