Mathematics 4K (TMA4120)

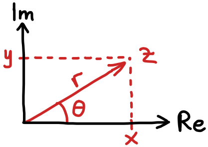
Parallel 2: MTTK

Lecture 14

Summary: Complex Analysis

1. Complex number:

$$z = x + iy = (x, y) = re^{i\theta} \qquad \boxed{i^2 = -1}$$



2. Complex exponential function:

$$\boxed{e^z = e^{x+iy} = e^x(\cos y + i \sin y)}$$

Extension of real exponential to $\mathbb{C}$

$2\pi i$-periodic: $e^{z+2\pi i} = e^z$

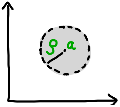
$e^{z_1} e^{z_2} = e^{z_1+z_2}$

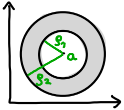
3. Roots: $w = \sqrt[n]{z} \Leftrightarrow w^n = z = re^{i\theta}$

$$\boxed{w = \sqrt[n]{r} e^{i\left(\frac{\theta}{n} + 2\pi \frac{k}{n}\right)}, \quad k = 0, 1, \dots, n-1}$$

4. Sets:

Circle:	$\lvert z-a \rvert = \rho$
Open disk:	$\lvert z-a \rvert < \rho$
Closed annulus:	$\rho_1 \leq \lvert z-a \rvert \leq \rho_2$
Half plane:	$\mathrm{Re}\, z > 0$, $\mathrm{Im}\, z \leq 0, \dots$





Lecture 14: Complex Analysis

Kreyszig: Section 13.3, 13.4

1. Sets: Open, connected, domains
2. Complex functions
3. Limits, continuity, derivative
4. Analytic functions, Cauchy-Riemann equations

Sets – as in $\mathbb{R}^2$

Limits, continuity – as for functions $f : \mathbb{R}^2 \to \mathbb{R}^2$

Derivatives – as for functions $f : \mathbb{R} \to \mathbb{R}$

OBS: You need 8 (av 12) exercises approved for taking the exam!!

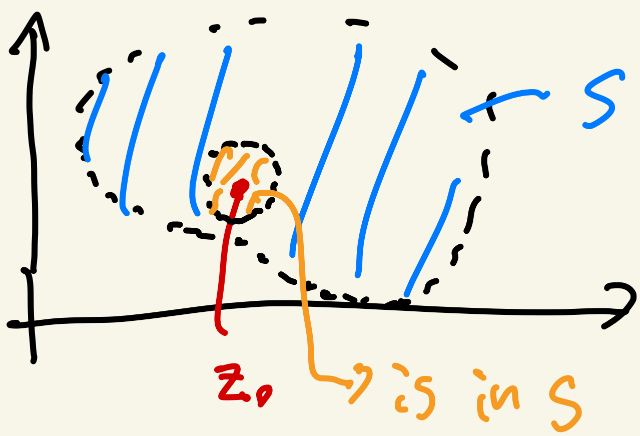
Lecture 14.

Concepts on $\mathbb{C}$

Definition. Let S a set in $\mathbb{C}$.

i) S is open if every point in S has a neighbourhood consisting entirely of S.

$\Big(z_0 \in S \Rightarrow$ There exists $\varepsilon > 0$ s.t.
$\{ z : |z - z_0| < \varepsilon \} \subset S \Big)$



ii) S is closed if the complement

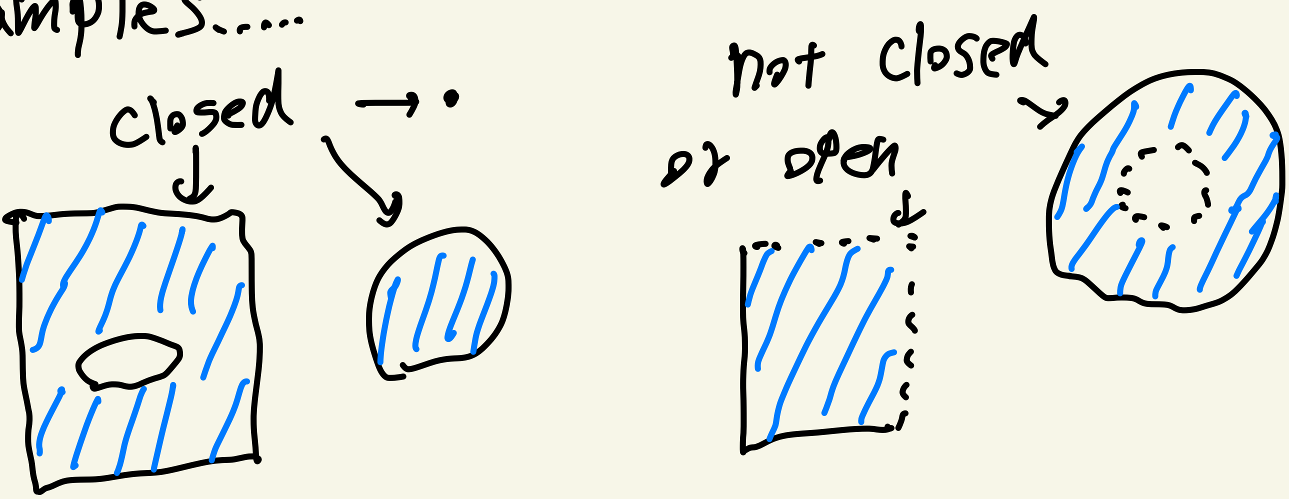
$S^c := \{ z \in \mathbb{C} : z \notin S \}$ is open.

iii) S is connected if any two points in S can always be joined by finitely long continuous curve which entirely lies in S.

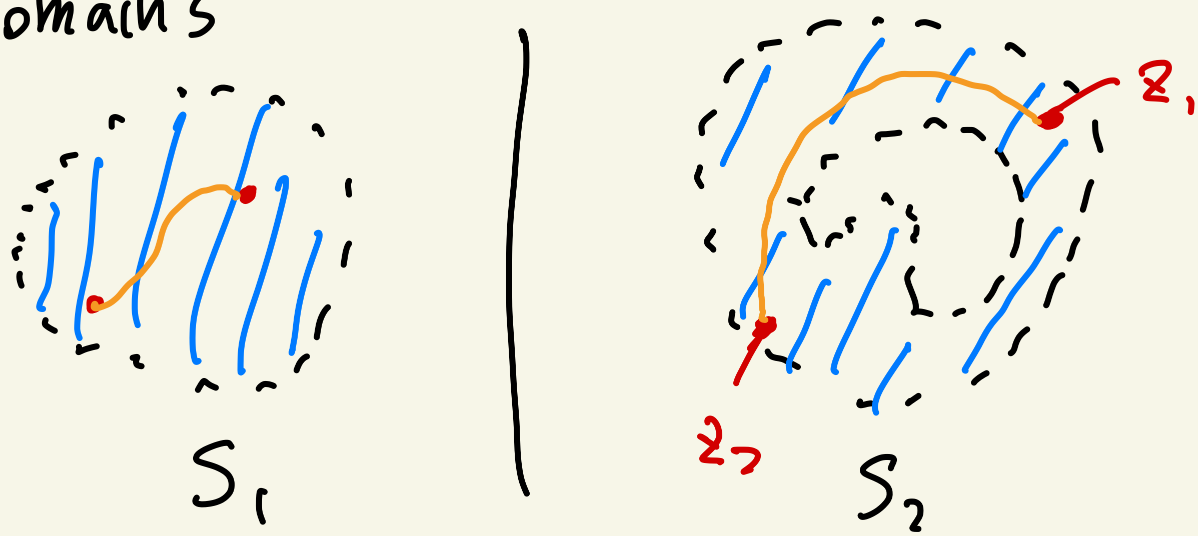
iv) S is domain if S is open and connected.

Examples.....

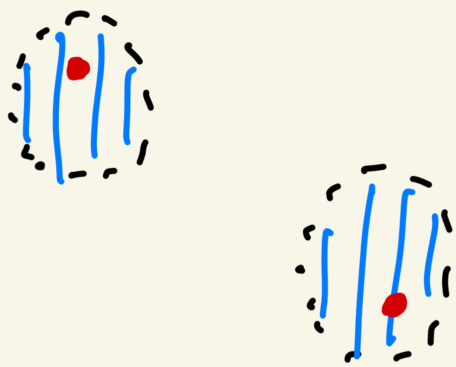
i)



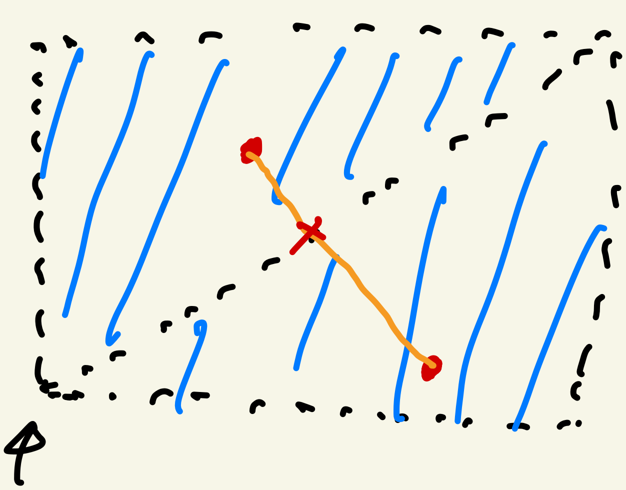
ii) domains



iii) not domains.....



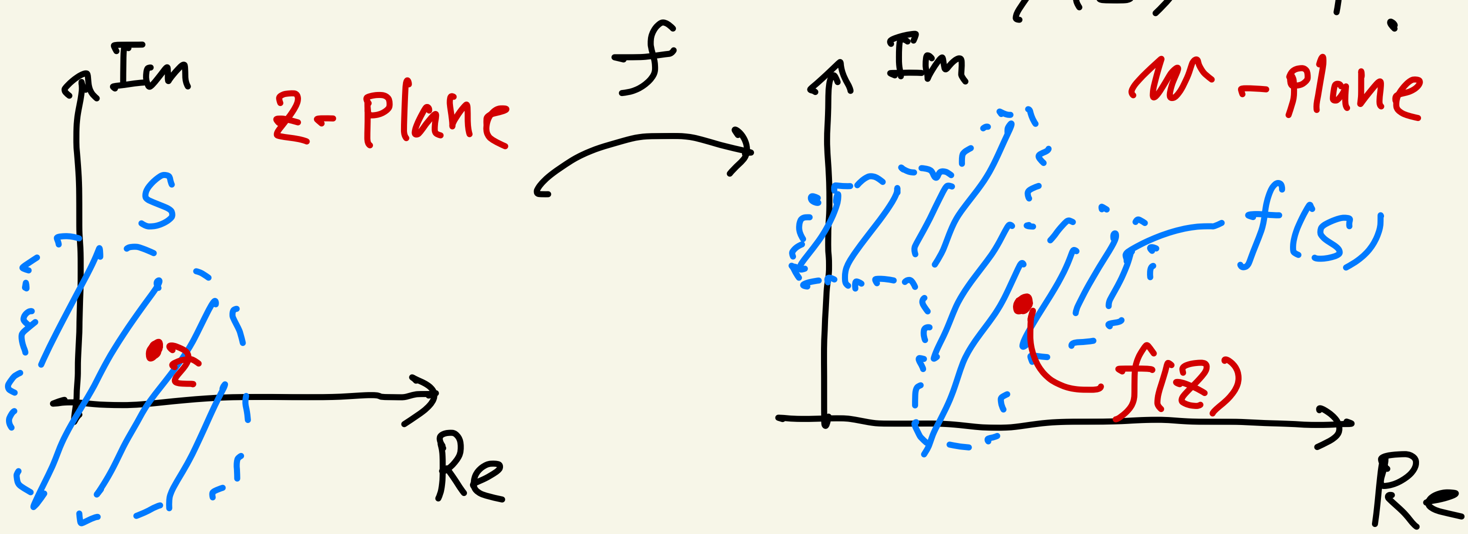
S_1



diagonal line is excluded

Complex functions

Definition A complex function is a rule/mapping which assigns for every $z \in S \subset \mathbb{C}$ a complex number $w = f(z) \in \mathbb{C}$.



Remark. By writing

$u = \mathrm{Re}(w)$, $v = \mathrm{Im}(w)$ $(u, v \in \mathbb{R})$

u, v are functions of $z = x + iy$

$\Rightarrow$ u and v are functions of x and y.

Thus, $w = f(z) = u(x,y) + i\,v(x,y)$

where $x, y \in \mathbb{R}$ and $\underbrace{u, v}_{\mathbb{R}^2 \to \mathbb{R}}$ are real functions.

Ex 1. Decompose

$w = u + iv \ldots$

i) $w = f(z) = z^2 - i\bar{z}$

$z = x + iy$

$= (x^2 + 2ixy - y^2) - i(x - iy)$

$= \underbrace{(x^2 - y^2 - y)}_{u(x,y)} + i\,\underbrace{(2xy - x)}_{v(x,y)}$

$= u(x,y) + i\,v(x,y)$.

$f(1+2i) = u(1,2) + i\,v(1,2) = -5 + 3i$ //

ii) $e^z = f(z)$. $e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y))$

$= \underbrace{e^x\cos(y)}_{=u(x,y)} + i\,\underbrace{e^x\sin(y)}_{=v(x,y)}$

$= u(x,y) + i\,v(x,y)$

$f(1+\frac{\pi}{2}i) = u(1,\frac{\pi}{2}) + i\,v(1,\frac{\pi}{2}) = 0 + i\cdot e.$

Limit. continuity

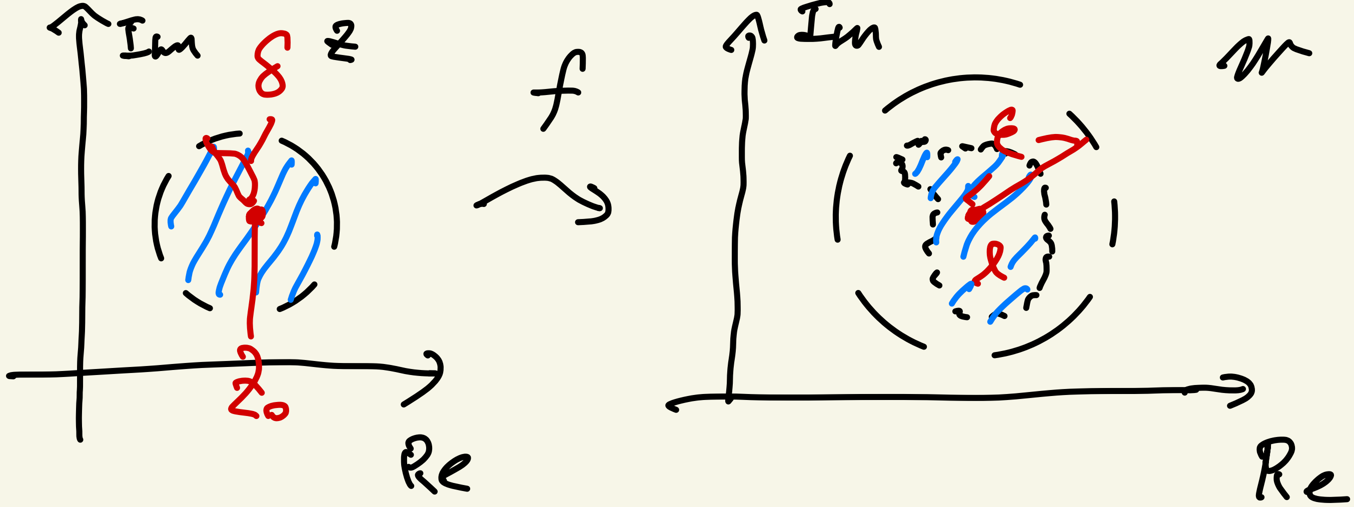
Definition (Limit)

$$\lim_{z\to z_0} f(z) = \ell \quad \left(\text{or } f(z) \xrightarrow[z\to z_0]{} \ell\right)$$

def
$\Longleftrightarrow$ For every $\epsilon > 0$, there exists $\delta > 0$

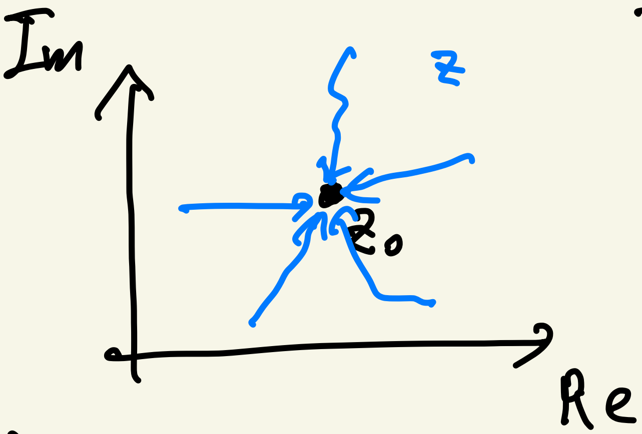
s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - \ell| < \epsilon$$



Remark.

i) "$z \to z_0$": all directions need to be considered



ii) $\lim_{z \to z_0} f(z)$ is defined when f is defined around the neighbourhood of z_0 (excluding z_0 itself)

Lemma 1.

a) $(\mathrm{Re}(z))^2 \leq |z|^2 = (\mathrm{Im}(z))^2 + (\mathrm{Re}(z))^2$

and $(\mathrm{Im}(z))^2 \leq |z|^2$

b) $|z - z_0| \to 0$

$\Leftrightarrow \mathrm{Re}(z - z_0) \to 0$ and $\mathrm{Im}(z - z_0) \to 0$.

proof) (a) $a^2 \leq a^2 + b^2$ (b) follows from (a) ☒

Ex 2. We want to show

$$\lim_{z \to z_0} f(z) = l$$

for $z_0 = i$, $l = 0$, $f(z) = z^2 + 1$.

$|z - i| < \delta$.

$\Rightarrow |f(z) - 0| = |z^2 + 1| = |z + i| \cdot |z - i|$

$(z^2 + 1) = (z + i)(z - i)$

$$= |z-i|\cdot|(z-i)+2i| \le |z-i|\cdot(|z-i|+|2i|)$$

$$< \delta\cdot(\delta+2) \xrightarrow[\delta\to 0]{} 0$$

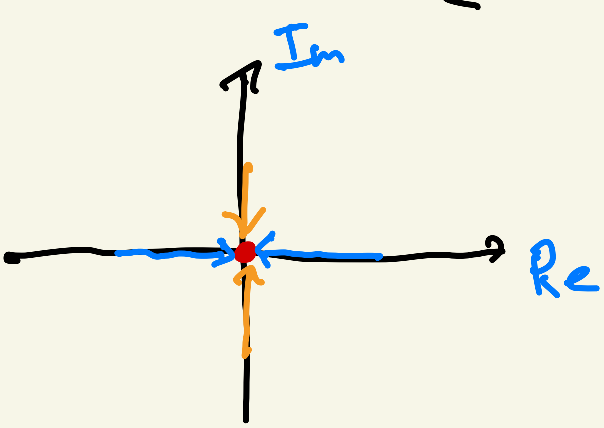
Therefore, for every $\varepsilon>0$, there exists $\delta>0$, s.t. $0<\delta(\delta+2)<\varepsilon$

(" $|z-i|<\delta \Rightarrow |f(z)-0|<\varepsilon$ " hold)

Thus $\displaystyle\lim_{z\to i}(z^2+1)=0$ //

Ex 3. $\boxed{f(z)\to ? \text{ as } z\to 0}$

$$f(z)=\frac{\bar z}{z}=\frac{x-iy}{x+iy} \begin{cases} \to 1 & \text{when } y=0,\ x\to 0 \\ \to -1 & \text{when } x=0,\ y\to 0 \end{cases}$$



$\Rightarrow \lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist!!

Definition [Continuity]

a) f is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$

b) f is continuous in domain $D \subset \mathbb{C}$, if

f is continuous every point $z_0 \in D$.

Derivatives

Definition [derivatives]

a)
$$f'(z) = \underbrace{\lim_{\Delta z \to 0}}_{} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

e limit from any direction on $\mathbb{C}$

b) f is differentiable at z_0 if

$f'(z_0)$ exists.

Ex4.

i) $(z^2)' = \lim\limits_{\Delta z \to 0} \dfrac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim\limits_{\Delta z \to 0} (2z + \Delta z)$

$= 2z$. (z^2 is differentiable everywhere!)

ii) $(\bar{z})' = \lim\limits_{\Delta z \to 0} \dfrac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$

$= \lim\limits_{\Delta z \to 0} \dfrac{\overline{\Delta z}}{\Delta z}$

This limit doesn't exist (see Ex.3).

$f(z) = \bar{z}$ is not differentiable anywhere.

Remarks

Differentiation rules are the same as real functions (proof are literally same)

$(fg)' = f'g + fg'$, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$(z^n)' = n z^{n-1}$, chain rules

(see page 623!)

Ex 5.

$f(z) = (iz+5)^4 + (2-\pi i)z^3 - (1+i)$

$f'(z) = 4 \cdot i(iz+5)^3 + 3(2-\pi i)z^2$ //

Analytic functions

Definition [analyticity]

a) $f(z)$ is analytic in a domain $D \subset \mathbb{C}$.

if f is defined and differentiable at every point $z_0 \in D$

b) $f(z)$ is analytic at a point z_0

if f is defined and differentiable in a neighbourhood of z_0 including z_0

Ex 6. Analytic functions.....

i) Polynomials...

$$f(z) = C_0 + C_1 z + C_2 z^2 + \cdots + C_n z^n$$

$\rightarrow$ analytic in $\mathbb{C}$

ii) Rational functions

$$f(z) = \frac{g(z)}{h(z)}$$ where g and h are Polynomials

$\rightarrow$ analytic except for where $h(z) = 0$.

iii) e^z is analytic in $\mathbb{C}$.

Cauchy-Riemann equation

Cauchy-Riemann eq. is given by

$$(1)\quad \boxed{u_x = v_y \quad \text{and} \quad u_y = -v_x}$$

Theorem 1.

"$w = f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D"

$\Longleftrightarrow$ "u_x, u_y, v_x, v_y all exist in D, and (1) is satisfied"

Ex 7. $f(z) = z^2$ is analytic in $\mathbb{C}$.

$$f(z) = \underbrace{(x^2 - y^2)}_{= u} + i \cdot \underbrace{2xy}_{v}$$

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x$$

Thus, $f(z) = z^2$ satisfies (1) in $\mathbb{C}$.

Ex 8.

$f(z) = \bar{z} = \underbrace{x}_{u} - i\underbrace{y}_{v=-y}$ is not analytic any where

Ex 5.

$u_x = 1, \quad v_y = -1, \quad u_y = -v_x = 0.$

$u_x \neq v_y.$

Thus, (1) is not satisfied any point in $\mathbb{C}$.

Ex 9. $f(z) = e^z = e^x(\cos(y) + i\sin(y))$

$= \underbrace{e^x\cos(y)}_{u} + i\,\underbrace{e^x\sin(y)}_{v.}$

$u_x = e^x\cos(y) = v_y$

$u_y = e^x(-\sin(y)) = -v_x$

Thus, e^z is analytic in $\mathbb{C}$.

Remark. If $f(z)$ is analytic everywhere in $\mathbb{C}$, f is said to be "entire function?

Lecture 14: Sets in $\mathbb{C}$

Open (contains neighborhood of each point)

Closed (complement open)

Connected (a curve connects any two points)

Domain (open, connected)

Lecture 14: Complex functions

A function f

a rule assigning each $z \in S$ a unique value $f(z) \in \mathbb{C}$

S: domain of definition

$$w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

Limits, continuity (as for function $\mathbb{R}^2 \to \mathbb{R}^2$)

Derivatives ($\approx$ as for functions $\mathbb{R} \to \mathbb{R}$)

- differentiation rules as for real functions

Lecture 14: Analytic functions

$f(z)$ analytic in domain D

if f defined and *differentiable* for all $z \in D$

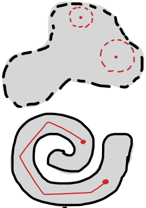
$$\Updownarrow$$

Cauchy-Riemann equations hold in D:

$$\boxed{u_x = v_y, \quad u_y = -v_x}$$

Summary: Complex Analysis

1. Sets in $\mathbb{C}$:

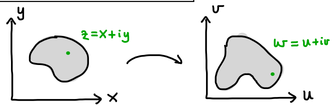
   

   Open: Contains open disk about each point
   Connected: Any two points can be connected
   by a finite continuous curve within the set
   Domain: Open and connected

2. Complex functions:

   Assigns each z in the domain of definition a unique value
   $f(z) \in \mathbb{C}$

   $$\boxed{w = f(z) = u(x,y) + iv(x,y)} \qquad \boxed{u = \operatorname{Re} w, \quad v = \operatorname{Im} w}$$

3. Limit, continuity: Same as for functions of 2 real variables
4. Derivative: Same definition/rules as for functions of one real variable
5. Analytic functions:

   $f(z)$ analytic in domain D if defined and *differentiable* in all $z \in D$
   $\Leftrightarrow$ Cauchy-Riemann equations hold in D: $u_x = v_y, \quad u_y = -v_x$