Mathematics 4K (TMA4120)

Parallel 2: MTTK

Lecture ~~12~~ 13

Summary: Partial differential equations

1. Concepts: Linear, homogeneous, order, solution,

Elliptic	Parabolic	Hyperbolic
Laplace $\Delta u = 0$	heat $u_t = \Delta u$	wave $u_{tt} = \Delta u$

Δ: Laplacian

$$\Delta := \frac{\partial^2}{\partial x^2} \left(+ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

2. Boundary value problems:

Cauchy	u given at $t = 0$
Dirichlet	u given on boundary
Neumann	(normal) derivative of u given on boundary

3. Solution methods (linear problems):

Separation of variables	$u_n = F_n(x)G_n(t)$ ... superposistion	rectangular domains
Fourier transform	transform - solve - invert	whole space
D'Alembert	change of variables	1D wave equation

4. Non-homogeneous: $u = u_h + u_p$, u_h homogeneous, u_p particular solution

Lecture 13: Complex Analysis

Kreyszig: Section 13.1, 13.2, 13.3, 13.5

1. Complex numbers
2. Complex exponential function
3. Polar form
4. Roots and equations

Most of this is repetition of Matematikk 3!!

Lecture 13

Ch. 13 Complex numbers and functions

Complex number

$z = x + iy, \quad x, y \in \mathbb{R}$.

$\cap$

$\mathbb{C}$ ← set of all complex numbers

Real part: $\mathrm{Re}(z) = x$

Imaginary part: $\mathrm{Im}(z) = y$.

Imaginary unit: i where $\boxed{i^2 = -1}$

Equality: $z_1 = z_2$

$$\iff \mathrm{Re}(z_1) = \mathrm{Re}(z_2)$$

and

$$\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$$

Calculation: We can calculate as if i is a real number using $i^2 = -1$.

Ex 1.

- $(3+5i)(2-3i) = 6 + 10i - 9i - 15i^2 = 21 + i$

$$\cdot\ \frac{3-5i}{2+2i} = \frac{3-5i}{2+2i}\,\frac{2-2i}{2-2i} = \frac{6-10i-6i+10i^2}{4-4i+4i-4i^2}$$

$$= \frac{-4-16i}{8} = -\frac{1}{2} - 2i$$

Remark:

Division: $\dfrac{z_1}{z_2} = \dfrac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \dfrac{z_1 \cdot \overline{z_2}}{|z_2|^2}$

Definition Let $z = x + iy$

a) complex conjugate: $\overline{z} = x - iy$.

b) Modulus / absolute Value: $|z| = \sqrt{x^2 + y^2} \geq 0$

c) Distance between z_1 and z_2

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Ex2.

$$\cdot\ \overline{\left(\frac{3-5i}{2+2i}\right)} \overset{\text{Ex.1}}{=} \overline{\left(-\frac{1}{2} - 2i\right)} = -\frac{1}{2} + 2i$$

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$$\frac{\overline{(3-5i)}}{\overline{(2+2i)}} = \frac{3+5i}{2-2i}\cdot\frac{2+2i}{2+2i} = \frac{6+6i+10i-10}{8}$$

$$= \underline{-\frac{1}{2}+2i}$$

- $$|(3-5i)-(8+8i)| = \sqrt{(3-8)^2+(-5-8)^2}$$

$$= \sqrt{194}.$$

Remarks.

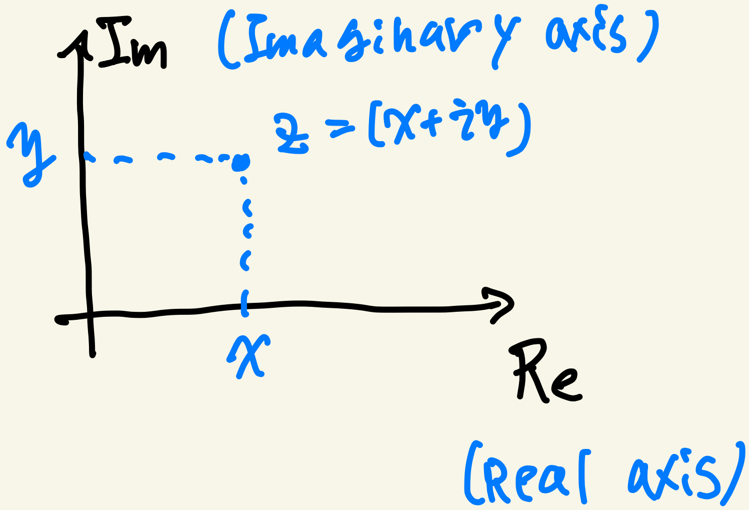
i) $\mathrm{Re}(z) = \frac{1}{2}(z+\overline{z})$, $\mathrm{Im}(z) = \frac{1}{2i}(z-\overline{z})$

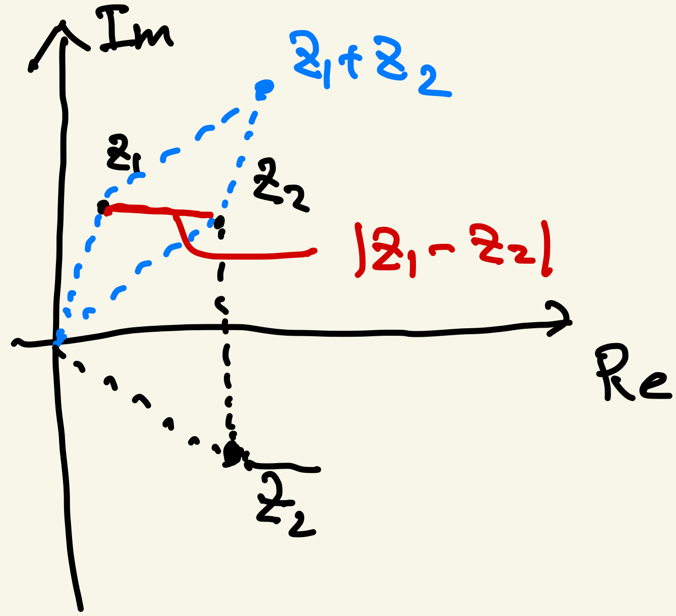
ii) $|z|^2 = z\cdot\overline{z} = x^2+y^2$

iii) $\overline{z_1+\frac{z_2}{z_3}} = \overline{z_1} + \frac{\overline{z_2}}{\overline{z_3}}$

iv) $z_1\cdot z_2 = z_2\cdot z_1$, $z_1(z_2+z_3) = z_1z_2+z_1z_3$.

v) The complex plane: we often plot $z\in\mathbb{C}$ on a plane.





Complex exponential function

Definition:

$$\boxed{e^{z} = e^{x+iy} = e^{x}(\cos(y)+i\sin(y))}$$

Remark.

i) When $z = x \in \mathbb{R}$, $e^{z} = e^{x+i\cdot 0} = e^{x}(1+0)$

$$= e^{x}$$

ii) When $z = iy$, $e^{z} = e^{iy} = \cos(y) + i\sin(y)$

iii) $2\pi i$ - Periodic

$$e^{z+2\pi i} = e^{x}(\overbrace{\cos(y+2\pi)}^{\cos(y)} + i\,\overbrace{\sin(y+2\pi)}^{\sin(y)})$$

$$= e^{z}$$

iv) Later we will see

$$(e^{z})' = e^{z}, \quad e^{z} = \sum_{n=0}^{\infty} \frac{z^{n}}{n!}$$

Theorem

$$\boxed{e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}}$$

$$\begin{cases} z_1 = x_1 + i y_1 \\ z_2 = x_2 + i y_2 \end{cases}$$

Proof)

$$e^{z_1+z_2} = e^{x_1+x_2}\Big(\underbrace{\cos(y_1+y_2)}_{=\cos(y_1)\cos(y_2) - \sin(y_1)\cdot\sin(y_2)} + \underbrace{i\sin(y_1+y_2)}_{=i\sin(y_1)\cos(y_2) + i\sin(y_2)\cos(y_1)}\Big)$$

$$= e^{x_1} e^{x_2} \big(\cos(y_1) + i\sin(y_1)\big)\big(\cos(y_2) + i\sin(y_2)\big)$$

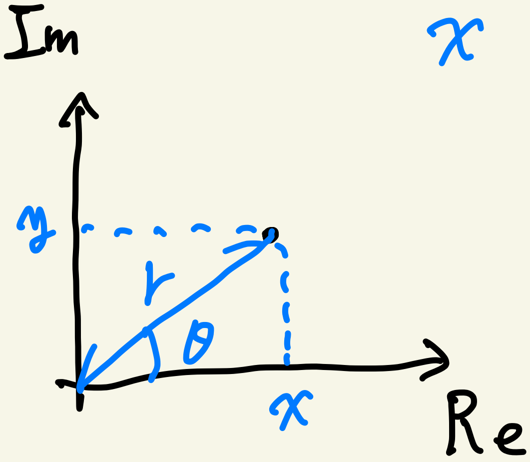
$$= e^{z_1} \cdot e^{z_2} \qquad \blacksquare$$

Polar form

Polar coordinate

$$x + iy \overset{\text{Polar coordinate}}{=} r\cos(\theta) + i\, r\sin(\theta) \overset{\text{def. of } e^{i\theta}}{=} r e^{i\theta}$$

x y



$r=\sqrt{x^2+y^2}$

θ: such that $\begin{cases} r\cos(\theta)=x \\ r\sin(\theta)=y. \end{cases}$

Remarks

i) $e^{i\theta}=e^{i(\theta+2\pi)}=e^{i(\theta+2n\pi)}$, $n\in\mathbb{Z}$.

ii) $r=|z|$

Definitions.

a) normal form: $z=x+iy$ (Cartesian form)

b) Polar form: $z=r\cdot e^{i\theta}$

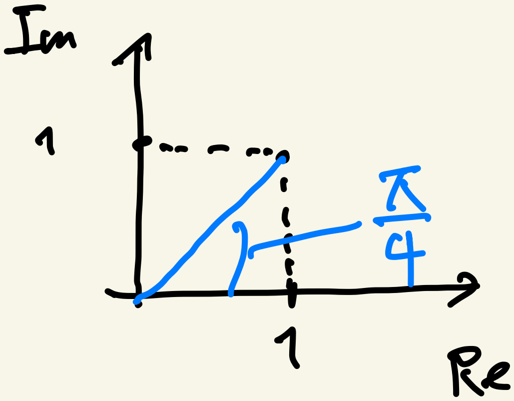
c) Argument/angle: $\arg(z)=\theta$

d) principal value of argument:

value θ, such that $-\pi<\theta\leq\pi$

Ex.3. Represent in the Polar form:

$z=1+i$, $\Rightarrow |z|=\sqrt{1^2+1^2}=\sqrt{2}$



$$\arg(z) = \frac{\pi}{4}.$$

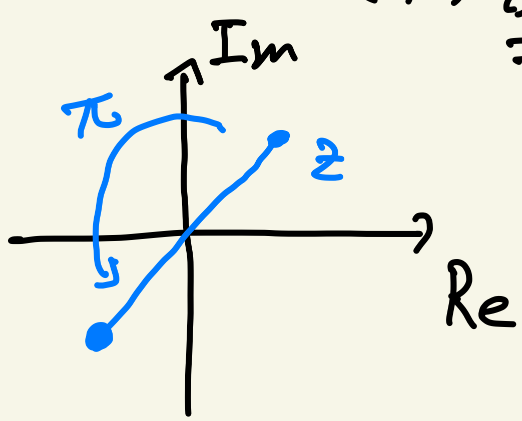
$$z = \sqrt{2} \cdot e^{i \cdot \frac{\pi}{4}}.$$

<u>Remark</u>: sometimes $\arctan\left(\frac{y}{x}\right)$ is used for finding $\arg(z)$, but....

$$\arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-y}{-x}\right)$$

$\Rightarrow$ same for z and $-z$

$\Rightarrow \arg(z) = \arctan\left(\frac{y}{x}\right)$ <u>or</u> $\arg(z) = \arctan\left(\frac{y}{x}\right) + \pi$



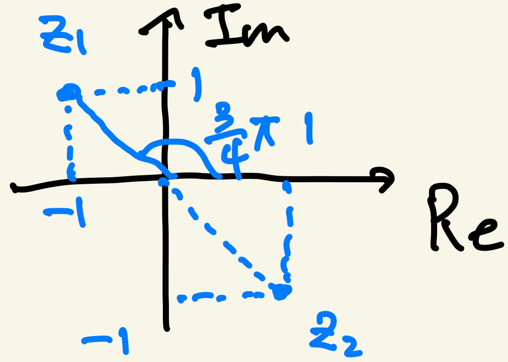
Advice: draw a graph (z in the complex plane)

<u>Ex4.</u> Represent in the Polar form:

$z_1 = -1 + i$

$z_2 = 1 - i$

$$|z_1| = |z_2| = \sqrt{1^2+1^2}$$
$$= \sqrt{2}$$



we want the principal value of arg.

$$\arg(z_1) = \frac{3}{4}\pi$$
$$\arg(z_2) = -\frac{1}{4}\pi.$$

$$\Rightarrow z_1 = \sqrt{2}\, e^{i\frac{3}{4}\pi}$$
$$z_2 = \sqrt{2}\, e^{-i\cdot\frac{1}{4}\pi}.$$

<u>Properties</u>. Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

i) $$\boxed{z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1+\theta_2)}}$$

$\Rightarrow |z_1 z_2| = |z_1| \cdot |z_2|$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

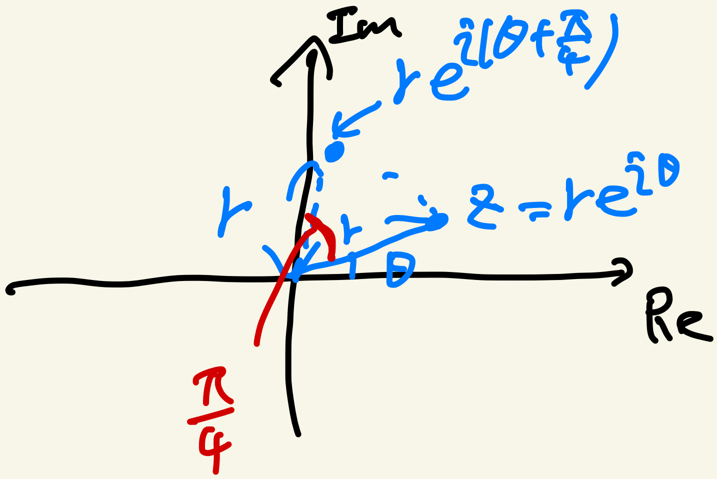
ii) $$\boxed{\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}}$$

$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

<u>Ex5.</u> Let $z = r e^{i\theta}$

$$z\cdot\underbrace{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)}_{=\,e^{i\frac{\pi}{4}}} = r\cdot e^{i(\theta+\frac{1}{4}\pi)}$$

multiplying $e^{i\frac{\pi}{4}}$ means rotation of $\frac{\pi}{4}$ on the complex plane



(cf. multiplying the matrix to a vector $\vec{a}\in\mathbb{R}^2$

$\begin{pmatrix}\cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4})\\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4})\end{pmatrix}$ had the exactly same effect.)

<u>Triangle inequality</u>: $|z_1+z_2|\le|z_1|+|z_2|$

<u>n-th root</u>.

Solve $w^n = z$ for w. $\left(\Leftrightarrow w = \underbrace{\sqrt[n]{z}}_{\text{mutivalued}}\right)$

write

$w = R \cdot e^{i\varphi}$, $\underline{z = r \cdot e^{i\theta}}$ known.

$\left(\begin{array}{l} x^2 = 1. \\ x = 1 \text{ or } x = -1 \end{array}\right)$

$w^n = z \Leftrightarrow R^n e^{in\varphi} = r \cdot e^{i\theta}$

$\Leftrightarrow R^n = r$, $n\varphi = \theta + 2k\pi$. $(k \in \mathbb{Z})$.

Thus. n distinct solutions are .....

$$\boxed{\sqrt[n]{z} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right)} \quad k = 0, \ldots, n-1}$$

Remarks

i) $k = k_0$, $k' = k_0 + n$, $k_0 \in \mathbb{Z}$.

$\Rightarrow$ We get the same number

(check, $e^{i\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right)}$)

is 2π-periodic

$\rightarrow$ n-periodic for k.

Ex6. Find $\sqrt[3]{1+i} = w$ for all w

$(\to w^3 = 1+i)$.

$$\text{Let } w = Re^{i\varphi}, \quad R^3 e^{i3\varphi} \overset{\text{Ex}}{=} \sqrt{2}\cdot e^{i\frac{\pi}{4}}.$$

$$\Rightarrow R^3 = \sqrt{2}, \ 3\varphi = \frac{\pi}{4} + 2k\pi \ (k \in \mathbb{Z}).$$

Thus, for $k = 0, 1, 2$, give us distinct solutions ....

$$w \in \left\{ \sqrt[6]{2}\cdot e^{i\frac{\pi}{12}}, \ \sqrt[6]{2}\, e^{i\frac{9}{12}\pi}, \ \sqrt[6]{2}\, e^{i\frac{17}{12}\pi} \right\}$$

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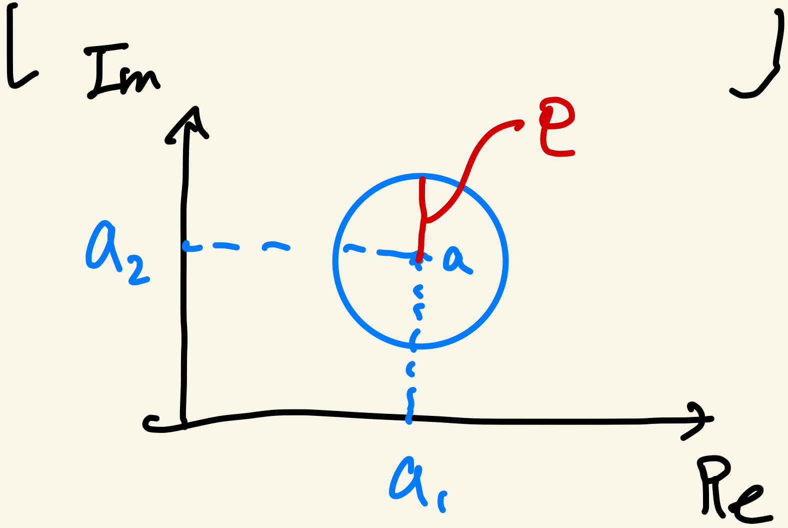
Circles, disks, and half-Plane on $\mathbb{C}$

Definition [sets]

(i) A circle of radius ρ ("rho") and center $a = a_1 + ia_2$, is given by $|z-a| = \rho$

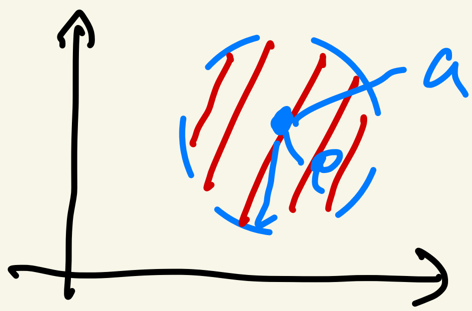
$$\{ z \in \mathbb{C} : |z-a| = \rho \}$$



(Note: $\rho^2 = |z-a|^2 \underset{\uparrow}{=} (x-a_1)^2 + (y-a_2)^2$)

$z = x + iy$

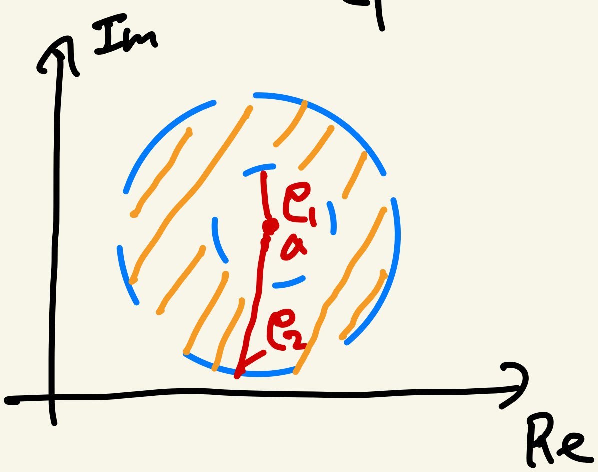
ii) open disk: $|z-a| < \rho$



iii) closed disk: $|z-a| \le \rho$

(Boundary is included!)

iv) open annulus: $\rho_1 < |z-a| < \rho_2$

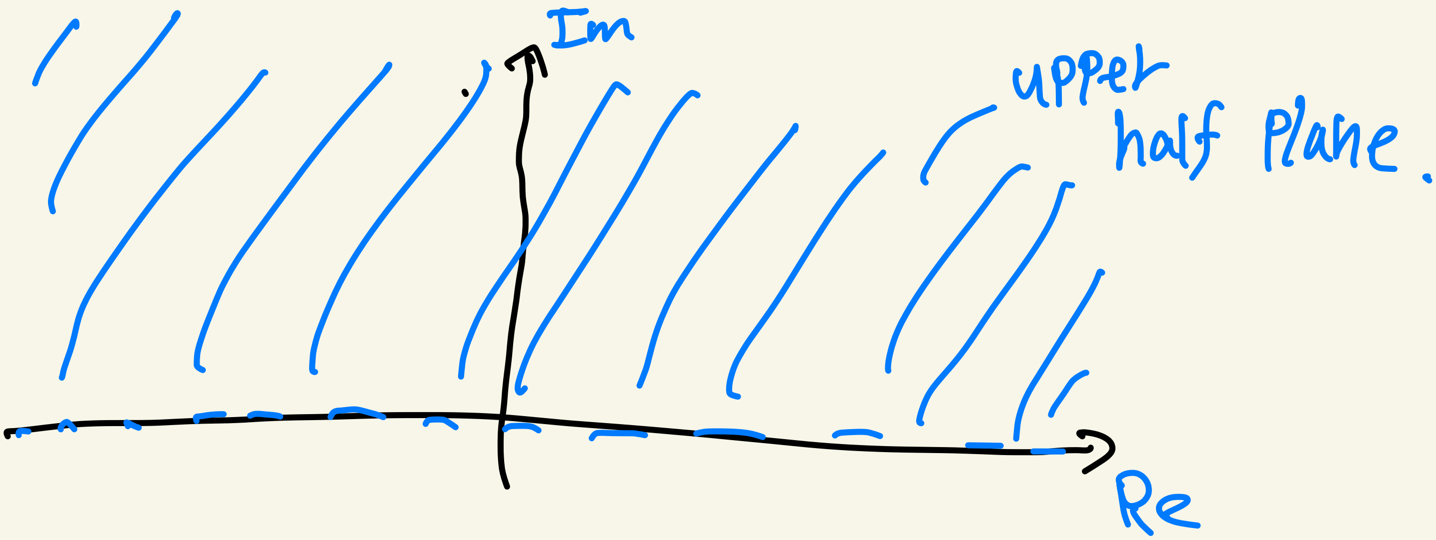


v) Closed annulus: $\rho_1 \le |z-a| \le \rho_2$

(the boundary included!)

vi) (open) upper, lower, right, left. half-plane.

$Im(z) > 0$, $Im(z) < 0$, $Re(z) > 0$, $Re(z) < 0$



Lecture 12: Complex numbers

$$\boxed{z = x + iy = re^{i\theta}} \quad \boxed{i^2 = -1} \qquad x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z),$$

$$\bar{z} = x - iy \qquad |z|^2 = z\bar{z} = x^2 + y^2 = r^2$$

$$r = |z|, \quad \theta = \arg(z) = \arctan(\tfrac{y}{x})\ (\pm\pi) \qquad \mathrm{Arg}(z) \in (-\pi, \pi]$$

Lecture 12: Complex exponential

$$\boxed{e^z = e^{x+iy} = e^x(\cos y + i \sin y)}$$

$$e^{z+2\pi i} = e^z \qquad (2\pi i\text{-periodic})$$

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

Lecture 12: Roots of complex numbers

$$w = \sqrt[n]{z}$$

$$\Leftrightarrow \quad w^n = z = re^{i\theta + i2\pi k} \quad (k \in \mathbb{Z})$$

$$\Leftrightarrow \quad w = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + 2\pi \frac{k}{n}\right)}, \quad k = 0, 1, \ldots, n-1$$

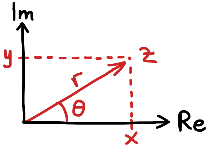
Lecture 12: Sets in $\mathbb{C}$

Circle:	$\lvert z-a\rvert=\rho$
Open disk:	$\lvert z-a\rvert<\rho$
Closed annulus:	$\rho_1 \le \lvert z-a\rvert \le \rho_2$
Half plane:	$\operatorname{Re} z>0$, $\operatorname{Im} z\le 0, \ldots$

Summary: Complex Analysis

1. Complex number:

$$z = x + iy = re^{i\theta} \qquad \boxed{i^2 = -1}$$



2. Complex exponential function:

$$\boxed{e^z = e^{x+iy} = e^x(\cos y + i \sin y)}$$

Extension of real exponential to $\mathbb{C}$

$2\pi i$-periodic: $e^{z+2\pi i} = e^z$

$e^{z_1} e^{z_2} = e^{z_1+z_2}$

3. Roots: $w = \sqrt[n]{z} \Leftrightarrow w^n = z$

$$\boxed{w = \sqrt[n]{r} e^{i(\frac{\theta}{n} + 2\pi \frac{k}{n})}, \quad k = 0, 1, \ldots, n-1}$$

4. Sets:

Circle:	$\lvert z-a \rvert = \rho$
Open disk:	$\lvert z-a \rvert < \rho$
Closed annulus:	$\rho_1 \le \lvert z-a \rvert \le \rho_2$
Half plane:	$\operatorname{Re} z > 0$, $\operatorname{Im} z \le 0, \ldots$