Mathematics 4K (TMA4120)

Parallel 2: MTTK

Lecture 12

Summary: Heat and Laplace equation

1. Boundary value problems:

   Cauchy u given at $t = 0$.
   Dirichlet u given on boundary
   Neumann (normal) derivative of u given on boundary

2. Heat equation:

$$
\begin{aligned}
&u_t = c^2 u_{xx} && t > 0,\ x \in (0, L) && (1)\\
&u(0, t) = 0 = u(L, t) && t > 0,\ x = 0, L && (2)\\
&u_x(0, t) = 0 = u_x(L, t) && t > 0,\ x = 0, L && (2')\\
&u(x, 0) = f(x) && t = 0,\ x \in (0, L) && (3)
\end{aligned}
$$

   u temperature of rod, ends: fixed temperature (2) or insulated $(2')$

   Cauchy-Dirichlet $(1), (2), (3)$; Cauchy-Neumann $(1), (2'), (3)$

3. Laplace equation: $u_{xx} + u_{yy} = 0$

   Electrostatic potiential, potiential flow, membrane, temperature ...

4. Solved by separation of variables, $u = F(x)G(t)$ ...

Review: Fourier transform

1. $$\boxed{\mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx}$$

2. $\mathcal{F}[f'](w) = (iw)\mathcal{F}[f](w)$

3. $\mathcal{F}[e^{-iax} f(x)](w) = \mathcal{F}[f](w + a)$

4. $\mathcal{F}^{-1}[e^{iaw} \hat{f}(w)](x) = f(x + a)$

5. $\mathcal{F}[f * g](w) = \sqrt{2\pi}\, \mathcal{F}[f](w) \cdot \mathcal{F}[g](w)$

   where $(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy$

6. $\mathcal{F}[e^{-ax^2}](w) = \dfrac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$

Lecture 12: Partial differential equations

Kreyszig: Section 12.4, 12.5, 12.7

1. PDEs:

   Heat equation (with derivation)

   Wave equation

   $x \in \mathbb{R}$

   $(-\infty < x < \infty)$

2. Solution techniques:

   Fourier transform

   Method of characteristics (D'Alembert)

Lecture 12.

Derivation of the heat equation

(12.5)

Temperature: u

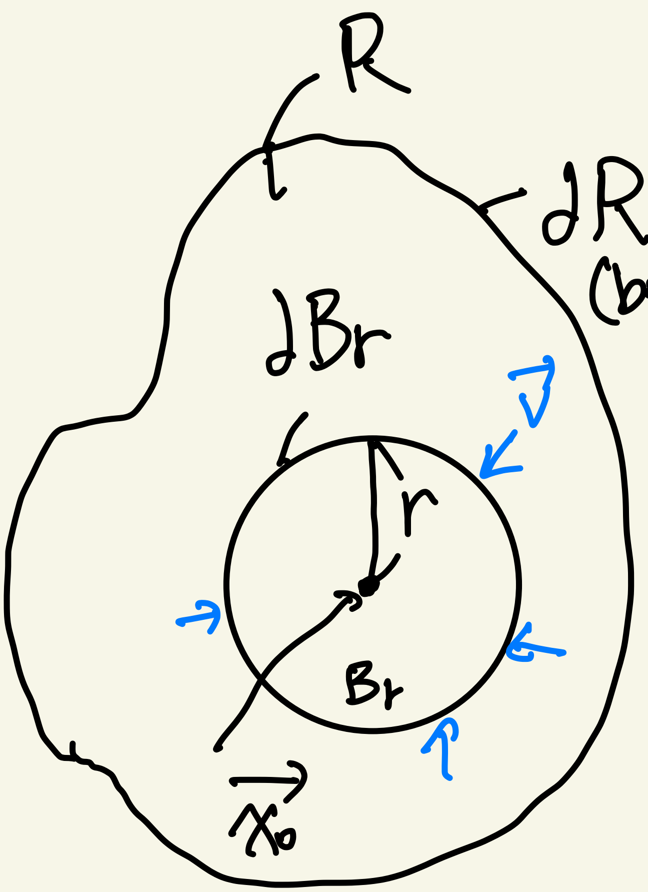
Heat Velocity: $\vec{v}$ $\left[\dfrac{\text{energy}}{\text{mass}\cdot\text{time}}\right]$

Specific heat capacity: σ $\left[\dfrac{\text{energy}}{\text{mass}\cdot\text{temp}}\right]$

Density: ρ $\left[\dfrac{\text{mass}}{\text{volume}}\right]$

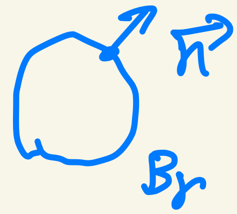
Fourier's law: $\vec{v} = -\underset{\substack{q \\ \text{constant}}}{\underline{k}} \cdot \nabla u$

$\left(\nabla u = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \right)$ in 3D case



Energy conservation in B_r:

Change/time = [flow in] − [flow out]



$$\frac{d}{dt}\iiint_{B_r} \overbrace{\sigma\rho u}^{\text{energy/volume}}\,dx\,dy\,dz = -\iint_{\partial B_r} \vec{V}\cdot\vec{n}\,ds$$

$\|$ $\qquad\qquad\qquad\qquad$ $\|$ divergence theorem

$$\iiint_{B_r} \sigma\rho u_t\,dx\,dy\,dz \qquad\qquad -\iiint_{B_r} \nabla\cdot\vec{V}\,dx\,dy\,dz.$$

$$\to \frac{1}{[\text{Volume}[B_r]]}\iiint_{B_r}\left(\sigma\rho u_t + \nabla\cdot\vec{V}\right)dx\,dy\,dz$$

$$= 0$$

$r \to 0$

$$\Rightarrow \sigma\rho u_t + \nabla \cdot \vec{V} = 0$$

for any $x = x_0$

Fourier's law

$$\Rightarrow u_t = \frac{-\nabla \cdot \vec{V}}{\sigma\rho} = \frac{k}{\sigma\rho} \cdot \nabla \cdot (\nabla u)$$

Thus.

$$\boxed{u_t = \frac{k}{\sigma\rho} \nabla \cdot (\nabla u) = c^2 (u_{xx} + u_{yy} + u_{zz})}$$

$c = \sqrt{\frac{k}{\sigma\rho}}$

This should hold for any $\vec{x}_0 \in \mathbb{R}^3$ and $t > 0$

Heat equation and Fourier transform

(infinitely long bar)

Consider the following Cauchy problem

(1) $\begin{cases} u_t = c^2 u_{xx}, & t>0,\ x\in\mathbb{R} \\ & (-\infty < x < \infty) \\ u(x,0) = f(x). \end{cases}$

we assume $f(x) \to 0$

as $x \to \pm\infty$.

solution).

<u>step 1.</u> Apply Fourier transform for variable x.

$$\mathcal{F}[u_t] \underset{\uparrow}{=} \frac{\partial}{\partial t}\mathcal{F}[u] = \hat{u}_t(w,t)$$

must be shown

("ikke pensum")

$$\mathcal{F}[u_{xx}] \underset{\uparrow}{=} -w^2\mathcal{F}[u] = -w^2\hat{u}(w,t)$$

rule for derivatives

also from the initial condition, we have,

$$\hat{u}(w,0) = \hat{f}(w).$$

and
$$\hat{u}_t(w,t) = -w^2 c^2 \hat{u}(w,t)$$

$$\left(\dot{V}_t = cV \rightarrow V = Ae^{ct} \right)$$

By solving $\hat{u}$,

$$\hat{u}(w,t) = \underbrace{\hat{u}(w,0)}_{\hat{f}(w)} \underbrace{e^{-c^2w^2t}}_{\hat{g}(w)}$$

<u>Step 2</u> Applying $\mathcal{F}^{-1}$, we have

$$g(x,t) = \mathcal{F}^{-1}\left[e^{-c^2w^2t}\right](x,t)$$

$$= \frac{1}{\sqrt{2c^2t}} e^{-\frac{x^2}{4c^2t}}$$

(recall

$$\mathcal{F}\left[e^{-ax^2}\right] = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$$
)

$$\Leftrightarrow \sqrt{2a}\, e^{-ax^2} = \mathcal{F}^{-1}\left[e^{-\frac{w^2}{4a}}\right]$$

in our case $a = \frac{1}{4c^2t}$

Thus,

$$u(x,t) = \mathcal{F}^{-1}[\hat{f}\cdot\hat{g}]$$

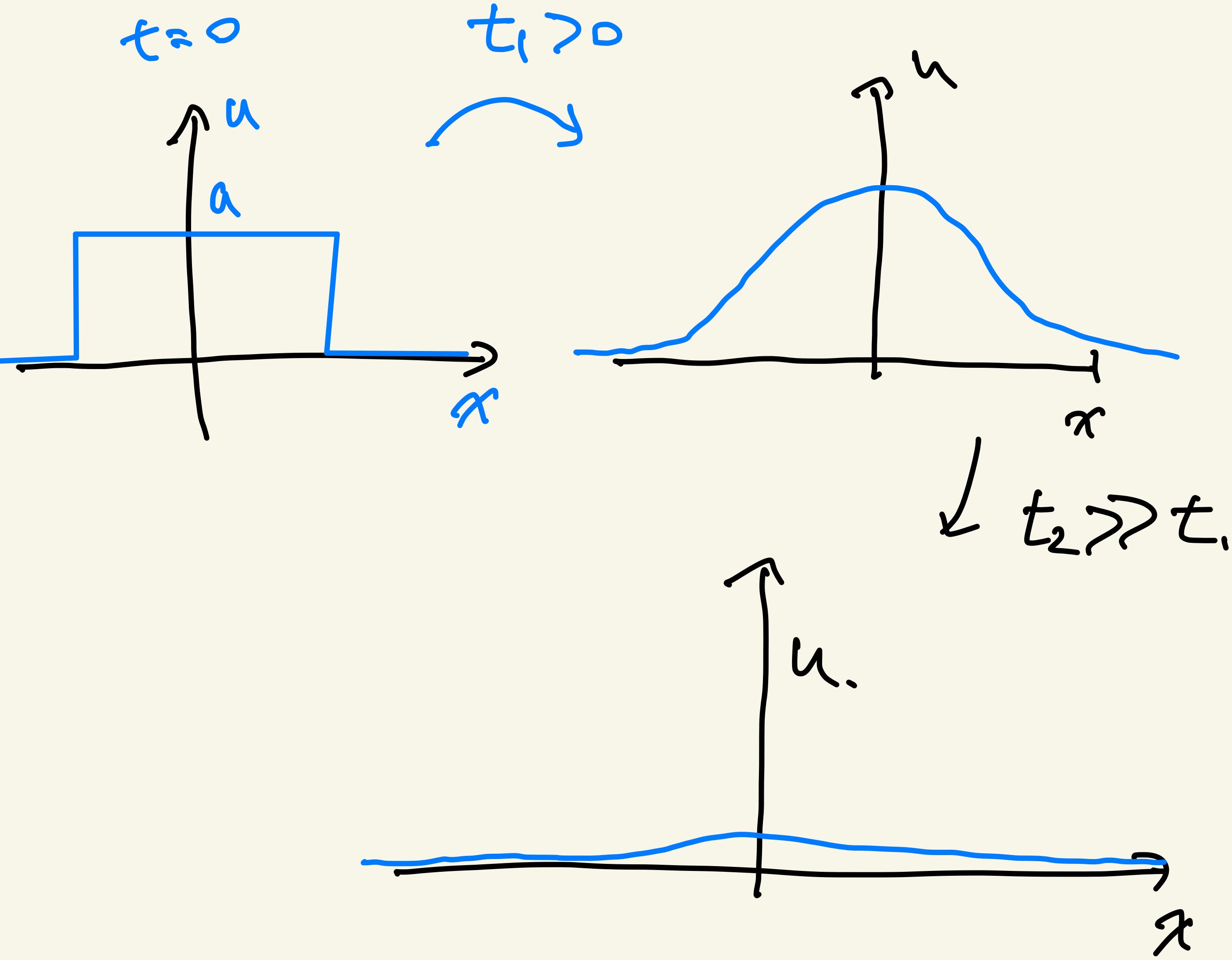
$$= \frac{1}{\sqrt{2\pi}}(f * g)(x,t)$$

Convolution theorem

$$= \frac{1}{\sqrt{4\pi c^2 t}} \int_{-\infty}^{\infty} f(y)\cdot e^{-\frac{(x-y)^2}{4c^2t}}\, dy.$$

<u>Ex 1</u>. Let $f(x) = \begin{cases} a, & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$$\Rightarrow u(x,t) = \frac{a}{\sqrt{4\pi c^2 t}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4c^2t}}\, dy.$$



Wave equation and Fourier transform.

$$
(2)\quad \begin{cases} u_{tt} = c^2 u_{xx}, & t>0,\ x\in\mathbb{R}. \\ u(x,0) = f(x) & x\in\mathbb{R} \\ u_t(x,0) = 0 & x\in\mathbb{R}. \end{cases}
$$

By applying Fourier transform,

We have

$$\begin{cases} \hat{u}_{tt} = -c^2 w^2 \hat{u} \\ \hat{u}(w,0) = \hat{f}(w) \\ \hat{u}_t(w,0) = 0 \end{cases}$$

Solving ODE for t

By solving the above eq....

$$\hat{u}(w,t) = A^{(w)} e^{icwt} + B^{(w)} e^{-icwt}$$

$$\hat{f}(w) = \hat{u}(w,0) = A^{(w)} + B^{(w)}.$$

and

$$A \cdot icw - Bicw = 0$$

$$\Rightarrow A = B = \frac{1}{2}\hat{f}(w)$$

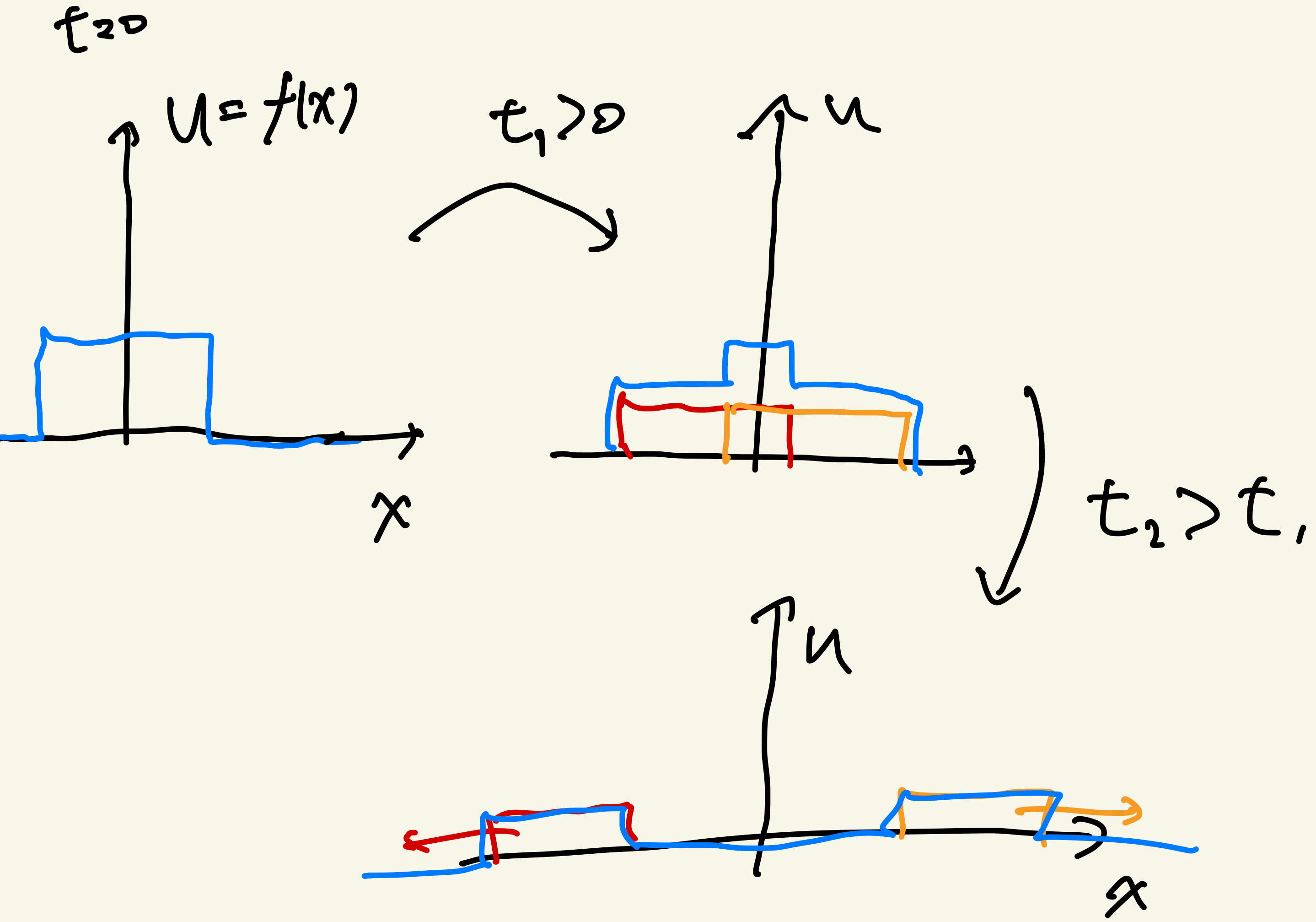
Thus,

$$\hat{u}(w,t) = \frac{1}{2}\hat{f}(w)\left(e^{icwt} + e^{-icwt}\right)$$

|| Applying $\mathcal{F}^{-1}$, use the

$\Downarrow$ shifting rule

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]} \quad (3)$$



Method of characteristics.

- Variables notation

$$(u, x, t) \longrightarrow (v, y, s)$$

$$u(x,t) = v(y(x,t), s(x,t))$$

- Choose η, s such that PDE for V becomes simple
- chacteristic curve: $\eta(x,t) = \text{const}$.
  $s(x,t) = \text{const}$
- only for hyperbolic equation!

$$u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}. \quad \text{— (4)}.$$

D'Alembert's Solution

Step1) Change of variables:

$$\eta = x + ct, \quad s = x - ct.$$

$$V(\eta, s) = V(x+ct, x-ct) = u(x,t)$$

Step2) Derive a simple PDE.

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial V(\eta, s)}{\partial x}$$

Chain rule

$$\stackrel{t}{=} \frac{\partial V}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial x}}_{=1} + \frac{\partial V}{\partial s}\cdot\underbrace{\frac{\partial s}{\partial x}}_{=1} = \frac{\partial V}{\partial y} + \frac{\partial V}{\partial s}$$

by differentiating again

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 V(y,s)}{\partial x^2}$$

$$= (V_{yy} + V_{sy})\underbrace{y_x}_{=1} + (V_{ys} + V_{ss})\underbrace{s_x}_{=1}$$

$$= V_{yy} + 2V_{ys} + V_{ss}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 V(y,s)}{\partial t^2} = \cdots$$

$$= c^2(V_{yy} - 2V_{ys} + V_{ss})$$

$(y_t = -s_t = c)$

By (4).

$$C^2(V_{yy} - 2V_{ys} + V_{ss}) = C^2(V_{yy} + 2V_{ys} + V_{ss})$$

$$\Leftrightarrow V_{ys} = 0 \quad \text{— (5)}$$

Step 3. Solve the simple PDE (5)

By integrating (5) ($\int_0^s (5)\, dr$)

$$0 = \int_0^s \frac{\partial}{\partial r}\left(\frac{\partial V(y, r)}{\partial y}\right) dr$$

$$\underset{\text{Fundamental thm. of Calculus}}{=} \frac{\partial V(y, s)}{\partial y} - \frac{\partial V}{\partial y}(y, 0) \quad \text{— (6)}$$

By integrating over y for (6)

$$0 = \int_0^y \left[\frac{\partial V(z, s)}{\partial z} - \frac{\partial V(z, 0)}{\partial z}\right] dz.$$

$$= \underbrace{V(y,s) - V(0,s)}_{\psi(s)\ \text{"psi"}} - \underbrace{\left(V(y,0) - V(0,0)\right)}_{\varphi(y)\ \text{"phi"}} \quad L(7)$$

Step 4). Conclusion.

$$\boxed{\begin{aligned} u(x,t) = V(y,s) &\overset{(7)}{=} \varphi(y) + \psi(s) \\ &= \varphi(x+ct) + \psi(x-ct) \end{aligned}} \quad (8)$$

Remarks

i) (8) solves (4) for all twice differentiable functions φ and ψ.

ii) All solutions of (4) are in the

form of (8).

<u>Ex3.</u> Cauchy problem

$$\begin{cases} (9) & u_{tt} = c^2 u_{xx} \\ (10) & u(x,0) = f(x) \\ (11) & u_t(x,0) = g(x) \end{cases}$$

<u>Solution.</u>

specify ψ and φ in (8) satisfying (10) and (11).

$$f(x) = u(x,0) = \varphi(x) + \psi(x) \quad \text{— (12)}$$

$$g(x) = u_t(x,0) = c\varphi'(x) \overset{-}{\cancel{+}} c\psi'(x)$$

$$\Leftrightarrow \frac{1}{c}\int_{-\infty}^{x} g(y)\,dy + \underbrace{K}_{\text{constant.}} = \varphi(x) - \psi(x) \quad \text{— (13)}$$

$$(12)+(13) \Rightarrow \varphi(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_{-\infty}^{x} g(y)\,dy + K\right)$$

$$(12)-(13) \Rightarrow \psi(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_{-\infty}^{x} g(y)\,dy - k\right)$$

thus,

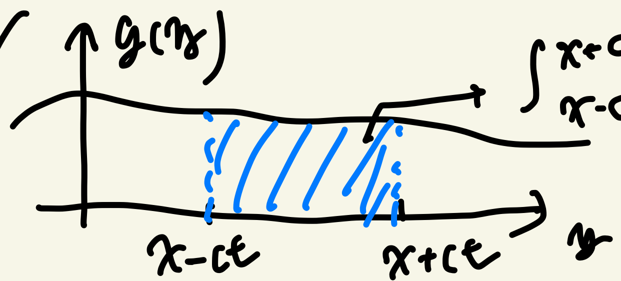
$$\boxed{\begin{aligned} u(x,t) &\overset{(5)}{=} \varphi(x+ct) + \psi(x-ct) \\ &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy \qquad \text{— (14)} \end{aligned}}$$

Remarks

i) (14) and $g(x) = 0$. gives (3)

ii) wave equation + two initial conditions

$\longrightarrow$ unique solution.



$$\left(\int_{x-ct}^{x+ct} g\,dy = \int_{-\infty}^{x+ct} g\,dy - \int_{-\infty}^{x-ct} g\,dy. \right)$$

Lecture 12: The heat equation in $\mathbb{R}$

$$\begin{cases} u_t = c^2 u_{xx} & t > 0,\ x \in \mathbb{R} \\ u(x,0) = f(x) & t = 0,\ x \in \mathbb{R} \end{cases} \tag{1}$$

1. Derivation

2. Solution using Fourier transform

(a) Fourier transform problem

$$\hat{u}_t = -c^2 w^2 \hat{u}, \quad \hat{u}(x,0) = \hat{f}(x)$$

(b) Solve

$$\hat{u}(w,t) = \hat{f}(w) e^{-c^2 w^2 t}$$

(c) Inverse Fourier transform

$$u(x,t) = (f * g)(x), \ g = \mathcal{F}^{-1}[e^{-c^2 w^2 t}]$$

Lecture 12: The wave equation in $\mathbb{R}$

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} & t > 0,\ x \in \mathbb{R} \qquad (3)\\
u(x,0) &= f(x) & t = 0,\ x \in \mathbb{R} \qquad (6)\\
u_t(x,0) &= g(x) & t = 0,\ x \in \mathbb{R} \qquad (7)
\end{aligned}
$$

1. Solution by Fourier transform

2. D'Alembert's solution

$$y = x + ct, \quad s = x - ct, \quad v(y,s) = u(x,t)$$

$$\Longrightarrow v_{ys} = 0 \underset{integrate}{\Longrightarrow} v(y,s) = \phi(y) + \psi(s)$$

$$\Longrightarrow \boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

Partial differential equations

1. Heat equation: $u_t = c^2 u_{xx}$

   Derived from Fourier's law and conservation of energy

2. Solution by Fourier transform:

$$
\begin{aligned}
& u_t = c^2 u_{xx}, \qquad u(x,0) = f(x) \\
\overset{\mathcal{F}}{\Longrightarrow} \quad & \hat{u}_t = -c^2 w^2 \hat{u}, \quad \hat{u}(w,0) = \hat{f}(w) \\
\overset{solve}{\Longrightarrow} \quad & \hat{u}(w,t) = \hat{f}(w) e^{-c^2 w^2 t} \\
\overset{\mathcal{F}^{-1}}{\Longrightarrow} \quad & u(x,t) = (f * g)(x,t) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy, \\
& \text{where } g(x) = \mathcal{F}^{-1}[e^{-c^2 w^2 t}](x) = \frac{1}{\sqrt{4\pi c^2 t}} e^{-\frac{x^2}{4c^2 t}}
\end{aligned}
$$

3. D'Alembert's solution of wave equation

$$
u_{tt} = c^2 u_{xx} \underset{\substack{y = x + ct, \\ s = x - ct, \\ v(y,s) = u(x,t)}}{\Longrightarrow} v_{ys} = 0 \underset{integrate}{\Longrightarrow} v(y,s) = \phi(y) + \psi(s)
$$

$$
\boxed{u(x,t) = v(x+ct, x-ct) = \phi(x+ct) + \psi(x-ct)}
$$