LAPLACE TRANSFORMS

Motivation

Laplace transforms make the process of solving differential equations much easier, since it is simplified to an algebraic problem. This technique of converting problems of calculus to algebraic problems is known as operational calculus. Laplace transforms find application in many areas of engineering and physics, e.g., electrical networks, springs, signal processing, etc.

**Def**

Let \( f(t) \) be a function of the real variable \( t \), defined for all \( t \geq 0 \). The Laplace transform of \( f \) is a function \( F(s) \) of the complex variable \( s \), denoted by \( \mathcal{L}(f) \), defined by the following relation:

\[
F(s) = \mathcal{L}(f) = \int_{0}^{\infty} e^{-st} f(t) \, dt
\]
Remark

If \( S = \alpha + i\beta \in \mathbb{C} \), then

\[
e^{-st}f(t) = e^{-\alpha t} \cos(\beta t) f(t) - ie^{-\alpha t} \sin(\beta t) f(t)
\]

We recall that the complex function \( e^z \), with \( z = x + iy \), is defined in terms of the real functions \( e^x \), \( \cos y \), and \( \sin y \) : \( e^z = e^x (\cos y + i \sin y) \).

Notice that in general also \( f(t) \) may be a complex function of the real variable \( t \), i.e. \( f(t) = f_1(t) + if_2(t) \) with \( f_1(t) \) and \( f_2(t) \) real functions.

The Laplace transform is an integral transform, i.e. it transforms a function in one space to a function in another space by a process of integration that involves a kernel, that is a function of the variables in the two spaces.

In this case:

\[
F(s) = \int_0^\infty K(s,t) f(t) \, dt
\]

with kernel function \( K(s,t) = e^{-st} \).
**Def** The given function in (1) is called inverse transform of $F(s)$ and is denoted by $\mathcal{L}^{-1}(F)$, i.e.

$$f(t) = \mathcal{L}^{-1}(F).$$

$$\Rightarrow \mathcal{L}^{-1}(\mathcal{L}(f)) = f \text{ and } \mathcal{L}(\mathcal{L}^{-1}(F)) = F$$

**Existence of Laplace transforms**

The integral in (1) is not meaningful for every function $f(t)$. Moreover, it is an improper integral dependent on the variable $s$ as a parameter and does not converge for all the values of $s$.

We give sufficient conditions for the existence of the Laplace transform in the following Theorem.

**Th. 1** If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies
the "growth restriction"

\[ |f(t)| \leq Me^{kt} \quad f(t) \text{ does not grow too fast!} \]

for all \( t \geq 0 \) and for some positive constants \( M \) and \( k \), then the Laplace transform \( \mathcal{L}(f) \) exists in the complex half-plane \( \text{Re}(s) > k \).

(for all \( s \geq k \) if \( s \) is real).

**Proof**

\( f(t) \) is piecewise continuous, i.e. of the form:

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Example of a piecewise continuous function \( f(t) \).
(The dots mark the function values at the jumps.)
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and therefore \( e^{-st}f(t) \) is integrable over any finite interval on the \( t \)-axis.

We have

\[
\left| \mathcal{L}(f) \right| = \left| \int_0^\infty e^{-st}f(t)\,dt \right| \leq \int_0^\infty |e^{-st}f(t)|\,dt
\]

\[
= \int_0^\infty e^{-\text{Re}(s)t} |f(t)|\,dt \leq \int_0^\infty \frac{M e^{-k\text{Re}(s)t}}{e^{-\text{Re}(s)t}}\,dt
\]

\[
= \lim_{T \to \infty} \int_0^T M e^{-k\text{Re}(s)t}\,dt = \frac{M}{k - \text{Re}(s)}
\]

The limit is 0 if \( k < \text{Re}(s) \).
In general, it can be proved that, if \( \mathcal{L}(f) \) exists for \( s = s_0 \), then it exists in the half-plane \( \{ s \in \mathbb{C} : \text{Re}(s) > \text{Re}(s_0) \} \) and \( \lim_{\text{Re}(s) \to \infty} \mathcal{L}(f) = 0 \).

Let \( \alpha_0 \) be the infimum of the set of numbers \( \alpha \) s.t. \( \mathcal{L}(f) \) exists for \( \text{Re}(s) > \alpha \). The half-plane \( \text{Re}(s) > \alpha_0 \) is called the region of convergence.

**Uniqueness**

If the Laplace transform of a given function exists, it is uniquely determined. Conversely, if two functions have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points (we say that the inverse of a given transform is essentially unique). If two continuous functions have the same transform, they are identical.
From now on we consider $s$ to be a real parameter for simplicity and ease of notation.

**Example 1**

$f(t) = 1$ for $t \geq 0$.

$L(f) = \int_{0}^{\infty} e^{-st} dt = \lim_{T \to \infty} \int_{0}^{T} e^{-st} dt = \lim_{T \to \infty} \left[ -\frac{1}{s} e^{-st} \right]_{0}^{T} = \lim_{T \to \infty} \left[ -\frac{1}{s} e^{-sT} + \frac{1}{s} e^{0} \right] = \lim_{T \to \infty} \left[ -\frac{1}{s} e^{-sT} \right] + \frac{1}{s} = \frac{1}{s}$

for $s > 0$.

**Example 2**

$f(t) = e^{at}$ for $t \geq 0$, a real constant.

$L(f) = \int_{0}^{\infty} e^{-st} e^{at} dt = \frac{1}{a-s} e^{-(s-a)t} \bigg|_{0}^{\infty} = \lim_{T \to \infty} \left[ -\frac{1}{a-s} e^{-(s-a)T} \right] - \frac{1}{a-s} e^{-(s-a)0} = \frac{1}{a-s}$.

$\Rightarrow L(f) = \frac{1}{s-a}$ when $s-a > 0$. 
Linearity of the Laplace transform.

The Laplace transform is a "linear operation", just as are differentiation and integration.

**Th. 2**

The Laplace transform is a linear operation, i.e., for any functions \( f(t) \) and \( g(t) \) whose transforms exist and any constants \( a \) and \( b \), the transforms \( af(t) + bg(t) \) exists and

\[
L \{ af(t) + bg(t) \} = aL \{ f(t) \} + bL \{ g(t) \}.
\]

**Proof**

It comes from the fact that integration is a linear operation, so we have

\[
L \{ af(t) + bg(t) \} = \int_0^\infty e^{-st} [af(t) + bg(t)] \, dt = \\
= a \int_0^\infty e^{-st} f(t) \, dt + b \int_0^\infty e^{-st} g(t) \, dt = \\
= aL \{ f(t) \} + bL \{ g(t) \}.
\]
EXAMPLE 3

\[ f(t) = \sinh(at) = \frac{e^{at} - e^{-at}}{2} \quad \text{hyperbolic sine} \]

\[ \mathcal{L}(f) = \frac{1}{2} \left[ \mathcal{L}(e^{at}) - \mathcal{L}(e^{-at}) \right] = \frac{1}{2} \left( \frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{a}{s^2 - a^2}. \]

EXAMPLE 4

\[ f_c(t) = \cos\omega t, \quad f_s(t) = \sin\omega t \]

\[ \mathcal{L}\{f_c(t)\} = \int_0^\infty e^{-st} \cos\omega t \, dt = (\text{by parts}) \]

\[ = \frac{e^{-st}}{-s} \cos\omega t \bigg|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin\omega t \, dt \]

\[ = \frac{1}{s} - \frac{\omega}{s} \mathcal{L}\{f_s(t)\}. \]

with the same procedure we arrive at

\[ \mathcal{L}\{f_s(t)\} = \frac{\omega}{s} \mathcal{L}\{f_c(t)\} \]

\[ \Rightarrow \mathcal{L}\{f_c(t)\} = \frac{1}{s} - \frac{\omega^2}{s^2} \mathcal{L}\{f_c(t)\} \]

\[ \mathcal{L}\{f_c(t)\} \left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s} \]

Check it by yourself!
\[ L \{ f_c(t) \} = \frac{s}{s^2 + \omega^2}. \]

Then
\[ L \{ f_s(t) \} = \frac{\omega}{s} L \{ f_c(t) \} = \]
\[ = \frac{\omega}{s} \frac{s}{s^2 + \omega^2} = \frac{\omega}{s^2 + \omega^2}. \]

\[ \text{EXAMPLE 5} \]

\[ f(t) = t^m \quad (m = 0, 1, \ldots) \]

Let us prove that
\[ L(f) = \frac{m!}{s^{m+1}} \]

By induction

\[ m = 0 \quad f(t) = 1, \quad L(1) = \frac{1}{s} \text{ TRUE (recall example)} \]

Let us suppose it is true for any \( m \geq 0 \) (induction hypothesis) and let us prove it for \( m+1 \).

\[ L(t^{m+1}) = \int_0^\infty e^{-st} t^{m+1} \, dt = (by \ points) \]
\[ = -\frac{1}{s} e^{-st} t^{m+1} \bigg|_0^\infty + \frac{m+1}{s} \int_0^\infty e^{-st} t^m \, dt \]

\[ \mathcal{L}(t^m) = \frac{m+1}{s} \]  
\[ \Rightarrow \]  
\[ \Delta \]  
Induction hypothesis

**Example 6**

\( f(t) = t^a \), a positive

\[ \mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a \, dt = \left( \frac{st \cdot x}{\frac{dx}{s}} \right) \]

\[ = \int_0^\infty e^{-x} \left( \frac{x}{s} \right)^a \frac{dx}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, dx \]

\[ := \frac{1}{s^{a+1}} \Gamma(a+1). \quad (s > 0) \]

\( \Gamma \) is the so called gamma function.
If we know \( \mathcal{L}\{f(t)\} \), we can immediately get \( \mathcal{L}\{e^{at}f(t)\} \).

**Th. 3**  
If \( f(t) \) has the transform \( F(s) \) (where \( s > K \) for some \( K \)), then \( e^{at}f(t) \) has transform \( F(s-a) \) (where \( s-a > K \)), that is:

\[
\mathcal{L}\{e^{at}f(t)\} = F(s-a)
\]
or

\[
e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}.
\]

**Proof.**

\[
F(s-a) = \int_0^\infty e^{-(s-a)t} f(t) \, dt = \int_0^\infty e^{-st} e^{at} f(t) \, dt = \mathcal{L}\{e^{at}f(t)\}.
\]

If \( F(s) \) exists (i.e., the integral converges) for \( s \) greater than some \( K \), then the first integral above exists for \( s-a > K \).
**EXAMPLE 7**

\[ f(t) = e^{at} \cos(\omega t) \]

Let 
\[ F(s) = \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2} \]

Then
\[ \mathcal{L}(f) = F(s-a) = \frac{s-a}{(s-a)^2 + \omega^2} \]

Similarly,
\[ \mathcal{L}\left(e^{at}\sin(\omega t)\right) = \frac{\omega}{(s-a)^2 + \omega^2} \]

**EXAMPLE 8**

Find the inverse transform of
\[ L(f) = \frac{3s - 137}{s^2 + 2s + 401} \]

We notice that
\[ \frac{3s - 137}{s^2 + 2s + 401} = \frac{3(s+1) - 140}{(s+1)^2 + 20^2} \]

\[ = 3 \frac{s+1}{(s+1)^2 + 20^2} - 7 \frac{20}{(s+1)^2 + 20^2} \]
The inverse transform \( \mathcal{L}^{-1} \) is linear, indeed given \( F(s) = \mathcal{L}\{f(t)\} \) and \( G(s) = \mathcal{L}\{g(t)\} \), we have

\[
\mathcal{L}^{-1}\left\{ aF + bG \right\} =
\]

\[
= \mathcal{L}^{-1}\left\{ a \mathcal{L}\{f\} + b \mathcal{L}\{g\} \right\} =
\]

\[
= \mathcal{L}^{-1}\left\{ \mathcal{L}\{af + bg\} \right\} =
\]

\[
af + bg .
\]

Therefore, going back to our example we get

\[
f = \mathcal{L}^{-1}\left\{ \frac{3s - 137}{s^2 + 2s + 201} \right\} =
\]

\[
= 3 \mathcal{L}^{-1}\left\{ \frac{s + 1}{(s+1)^2 + 20^2} \right\} - \mathcal{L}^{-1}\left\{ \frac{20}{(s+1)^2 + 20^2} \right\} =
\]

\[
e^{-t}\left[ 3 \cos(20t) - 7 \sin(20t) \right] .
\]

\[\text{damped vibration}\]

\[\begin{array}{c}
0.5 \quad 1.0 \quad 1.5 \quad 2.0 \quad 2.5 \quad 3.0 \quad t \\
-6 \quad -4 \quad 2 \quad 4 \quad 6
\end{array}\]