Complex numbers

Motivation

Even the most elementary algebraic operations involving real numbers take us beyond the domain of rela numbers. For example, not every algebraic equation can be solved in terms of real numbers (eg. $x^2 + 1 = 0$). This led to the introduction of complex numbers, that extend the domain of real numbers so that basic algebraic operations can always be employed. In particular, exponentiation can be defined in general (while on the real field the operation a^b is defined for every exponent b if the base a is positive, and only for integer or rational exponents m/n, with n odd, if the base is negative).
Moving into the domain of complex numbers, denoted by $\mathbb{C}$, and introducing functions of complex variables becomes convenient in many cases, for example when integrating elementary functions or when solving differential equations. The complex notation is also convenient in the mathematical formulation of many physical propositions (e.g. in electrical engineering).

Complex numbers and operations on complex numbers

A complex number z is characterized by a pair of real numbers (x, y), having established a sequential order of the

numbers x and y. This is written as $z = (x, y)$.
x is called the real part and y the imaginary part of z, and we write

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z).$$

$z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal, i.e $z_1 = z_2$, if and only if $x_1 = x_2$ and $y_1 = y_2$.

Algebraic operations including complex numbers

The sum of $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined by

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

It is easy to check that in this definiton the commutative $\Big(z_1 + z_2 = z_2 + z_1\Big)$ and associative $\Big(z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3\Big)$ laws of addition hold true.

The product is defined by

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

It is easy to check that in this definiton the commutative $\Big(z_1 z_2 = z_2 z_1\Big)$, associative $\Big(z_1(z_2 z_3) = (z_1 z_2) z_3\Big)$ and distributive $\Big((z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3\Big)$ laws of multiplication hold true.

We can include the real numbers in the set of complex numbers by regarding the real number a as the complex number $a = (a, 0)$. We have $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$ as for real numbers. This is an example showing how the set of complex numbers is an extension of the set of real numbers.

Notice that the set of complex numbers, unlike the set of real numbers, does not possess the property of ordering (i.e. $z_1 > z_2$ or $z_1 < z_2$ have no meaning for complex numbers!).

$z = (0, b)$ is called a pure imaginary number ad is symbolized as $z = ib$. It can be regarded as the product of the imaginary unit $(0, 1)$, denoted by the symbol i (and sometimes by j), and a real number $(b, 0)$. By the definition of product we have $i \cdot i = i^2 = -1$.

The previous relation enables us to introduce the algebraic form (or Cartesian form) of a complex number,

$$z = (x, y) = x + iy,$$

and perform operations of addition and multiplication of complex numbers with the usual rules of algebra of polynomials.

$z = x + iy$ is termed difference between $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, i.e.

$$z = z_1 - z_2,$$

if $z_1 = z + z_2$, hence $x = x_1 - x_2$ and $y = y_1 - y_2$.

$z = x + iy$ is called the quotient of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, i.e.

$$z = \frac{z_1}{z_2} \quad (z_2 \neq 0),$$

if $z_1 = zz_2$, hence

$$\begin{cases} x_2 x - y_2 y = x_1, \\ y_2 x + x_2 y = y_1. \end{cases}$$

We get

$$z = \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}.$$

A practical rule to perform the above division is the following:

$$\begin{aligned} z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{x_1 x_2 - ix_1 y_2 + iy_1 x_2 - i^2 y_1 y_2}{x_2^2 - ix_2 y_2 + ix_2 y_2 - i^2 y_2^2} \\ &= \frac{x_1 x_2 - ix_1 y_2 + iy_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}. \end{aligned}$$

$\bar{z} = x - iy$ is said to be the complex conjugate number of $z = x + iy$.

Some useful relations involing complex conjugate numbers:

- $z\bar{z} = x^2 + y^2$

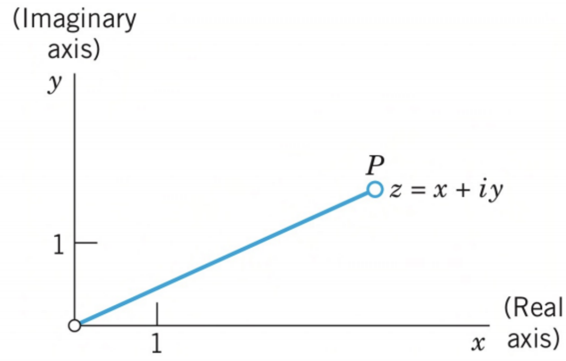
- $z + \bar{z} = 2x \Rightarrow \mathrm{Re}(z) = x = \frac{1}{2}(z + \bar{z})$
- $z - \bar{z} = 2iy \Rightarrow \mathrm{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$
- $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$
- $\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2}$
- $\overline{(z_1 z_2)} = \overline{z_1 z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

The geometric interpretation of complex numbers

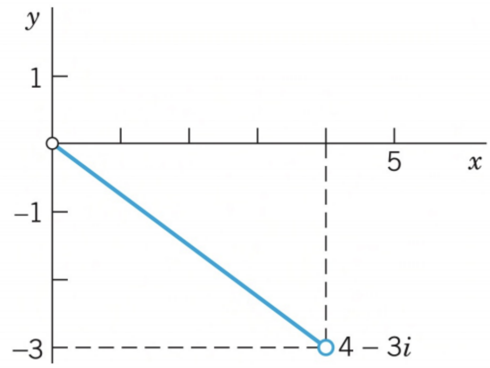
Being defined by a pair of real numbers, it is natural to represent the complex numbers in the plane. If we fix a Cartesian coordinate system, we let the complex number $z = a + ib$ correpond to the point with Cartesian coordinates (a, b). The number $z = 0$ corresponds to the origin of the plane. In this context, the plane is called the complex plane (or Gauss plane, or Argand diagram). The axis of abscissas is the real axis, the axis of ordinates is the imaginary axis.

In this way we have established a reciprocal one-to-one correpondence between the set of all complex numbers and the set of points in the complex plane, and also between the set of all complex numbers $z = a + ib$ and the set of free vectors, whose projections on the axis of abscissas and the axis of ordinates are, respectively, equal to a and b.

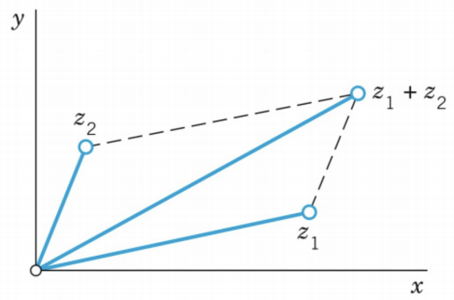
The correspondence between the set of all complex numbers and the plane vectors enables us to identify the operations of addition and subtraction of complex numbers with the corresponding operations involving vectors.



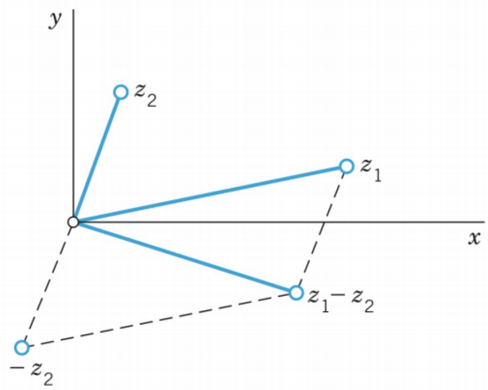
The complex plane



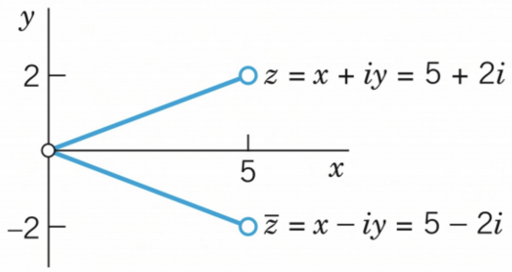
The number 4 − 3*i* in the complex plane



Addition of complex numbers



Subtraction of complex numbers



Complex conjugate numbers

Polar form of complex numbers

Taking advantage of the relationship between the Cartesian coordinates (x, y) and the polar coordinates (ρ, θ), defined by

$$x = \rho\cos(\theta), \quad y = \rho\sin(\theta),$$

we get the so-called trigonometric or polar form of a complex number:

$$z = \rho\,(\cos(\theta) + i\sin(\theta)).$$

ρ is called the modulus (sometimes magnitude or absolute value) of z and is denoted by $|z|$. We have

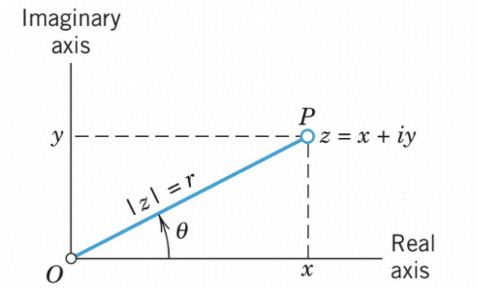
$$\rho = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Geometrically, $|z|$ is the distance of the point z from the origin. Similarly, $|z_1 - z_2|$ is the distance bewteen z_1 and z_2.

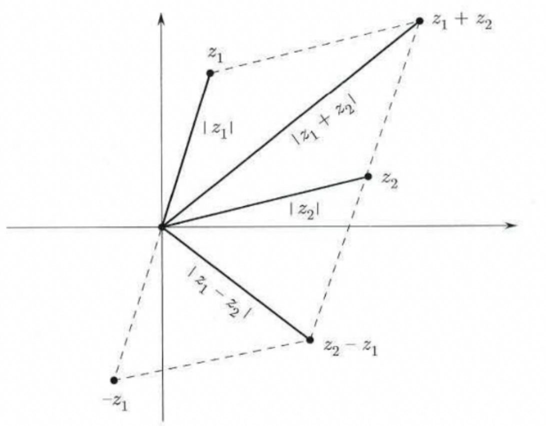
θ is called the argument (or also the phase) of z and is denoted by $\arg(z)$, i.e. $\theta = \arg(z)$. We have

$$\tan(\theta) = \frac{y}{x}.$$

Geometrically, θ is the angle that the radius vector of the given point makes with the positive direction of the axis of abscissas. Angles are usually measured in radians (sometimes also in degrees) and positive in the counterclockwise sense.



Polar form of a complex number



The argument of the complex number $z = 0$ is not defined and its modulus is zero.

For $z \neq 0$, the angle θ is not defined uniquely, but only up to integer multiples of 2π (due to the periodicity of sine and cosine). Since we often want to specify a unique value of $\arg(z)$, we define the principal value $\mathrm{Arg}(z)$ of $\arg(z)$ by the double inequality

$$-\pi < \mathrm{Arg}(z) \leq \pi \ .$$

We have that

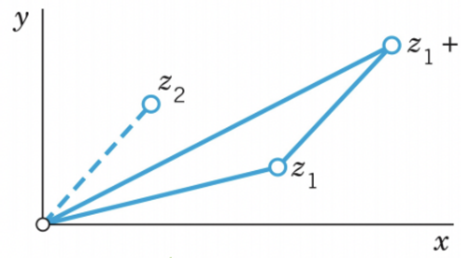
- $\mathrm{Arg}(z) = 0$ for positive real numbers;
- $\mathrm{Arg}(z) = \pi$ for negative real numbers;
- $\arg(z) = \mathrm{Arg}(z) \pm 2n\pi \quad (n = 1, 2, \ldots)$.

Note that we can specify a unique value of $\arg(z)$ in infinite ways, by fixing an interval of amplitude 2π in which θ can vary. Another interval commonly used to define the principal value is $[0, 2\pi)$. To make an example, the number $-i$ has argument $-pi/2 + 2k\pi$, with $k \in \mathbb{Z}$, and had principal value $-\pi/2$ with the convention $\theta \in (-\pi, pi]$ or $3\pi/2$ with the convention $\theta \in [0, 2\pi)$.

Triangle inequality

We recall that there is no natural way of ordering complex numbers (therefore, inequalities such as $x_1 < x_2$ make sense for real numbers, but not for complex numbers). However, inequalities between absolute values (which are real!), such as $|z_1| < |z_2|$, are of great importance. In particular, the following triangle inequalities hold:

$$|z_1 + z_2| \leq |z_1| + |z_2|\,, \qquad |z_1 - z_2| \geq |z_1| - |z_2|\,.$$



Geometric meaning of the triangle inequality

By induction:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Multiplication and division in polar form

The polar form comes in very handy when performing multiplication and division of complex numbers.

Let $z_1 = \rho_1 \left(\cos\theta_1 + i\,\sin\theta_1\right), \;\; z_2 = \rho_2 \left(\cos\theta_2 + i\,\sin\theta_2\right)$. Then

$$z_1 z_2 = \rho_1 \rho_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i\left(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta\right.\right.$$

By applying the addition rules for cosine and sine, we get

$$z_1 z_2 = \rho_1 \rho_2 \left[(\cos(\theta_1 + \theta_2) + i\,\sin(\theta_1 + \theta_2)\right].$$

We notice immediately that

$$|z_1 z_2| = |z_1||z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

Now let us consider the division. From

$$z_1 = \frac{z_1}{z_2} z_2,$$

we have

$$|z_1| = \left|\frac{z_1}{z_2}\right| |z_2| \quad \Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},$$

and

$$\arg(z_1) = \arg\left[\left(\frac{z_1}{z_2}\right) z_2\right] = \arg\left(\frac{z_1}{z_2}\right) + \arg(z_2) \quad \Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right)$$

Therefore,

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \left[(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right].$$

Integer powers of z. De Moivre's formula.

From the rule of multiplication in polar form, with $z_1 = z_2 = z$, by induction for $n = 0, 1, 2, \ldots$, we get

$$z^n = \rho^n \left[\cos(n\theta) + i \sin(n\theta)\right].$$

For $|z| = \rho = 1$ we obtain De Moivre's formula:

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$$

With this formula we can get $\cos(n\theta)$ and $\sin(n\theta)$ in terms of powers of $\cos(\theta)$ and $\sin(\theta)$. For example, for n=2:

$$(\cos(\theta) + i \sin(\theta))^2 = \cos^2(\theta) + \sin^2(\theta) + 2\cos(\theta)\sin(\theta).$$

But from De Moivre's formula we have

$$(\cos(\theta) + i \sin(\theta))^2 = \cos(2\theta) + i \sin(2\theta) \,.$$

Therefore,

$$\cos(2\theta) = \cos^2(\theta) + \sin^2(\theta), \qquad \sin(2\theta) = 2\cos(\theta)\sin(\theta) \,.$$

Exponential form of a complex number

Taking advantage of the following Euler's formula,

$$\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta),$$

we obtain the so called exponential form of a complex number:

$$z = \rho \mathrm{e}^{i\theta} \,.$$

In particular, $\mathrm{e}^{i\theta} = -1 \quad \Rightarrow \quad \mathrm{e}^{i\theta} + 1 = 0$ (the famous Euler's identity).
For the moment, consider the Euler's formula as an abridged form of representing the complex number $z = \cos(\theta) + i\sin(\theta)$
.

For the multiplication and division of complex numbers, we have:

$$z_1 z_2 = \rho_1 \rho_2 \mathrm{e}^{i(\theta_1 + \theta_2)} \quad , \qquad \frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} \mathrm{e}^{i(\theta_1 - \theta_2)} \,.$$

Roots

Let $z = r(\cos\theta + i\,\sin\theta)$ and let us compute $w = \sqrt[n]{z}$.
We write

$$w = R(\cos\phi + i\,\sin\phi)\,.$$

The equation $w = \sqrt[n]{z}$ that we want to solve is equal to

$$w^n = z$$

and by using De Moivre's formula we have

$$w^n = R^n\Big[\cos(n\phi) + i\,\sin(n\phi)\Big] = z = r(\cos\theta + i\,\sin\theta)\,.$$

Absolute values and arguments on both sides must be equal for this equation to hold.
For the absolute value we get

$$R^n = r \quad \Rightarrow \quad R = \sqrt[n]{r}$$

so that $\sqrt[n]{r}$ is a uniquely determined positive real number.
When equating the arguments $n\phi$ and θ, we have to remember that θ is determined only up to integer multiples of 2π, therefore

$$n\phi = \theta + 2\pi k \quad \Rightarrow \quad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

with k integer. For $k = 0, 1, \ldots, n-1$ we get n distinct values of w (indeed, for $k = n$ we get the same value of $k = 0$, for $k = n+1$ we get the same value of $k = 1$, and so on...), given by:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i \sin\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

We say that $w = \sqrt[n]{z}$ is multivalued, namely n-valued. The n distinct values of w lie on a circle of radius $\sqrt[n]{r}$ with centre at the origin and constitute the vertices of a regular polygon of n sides.

The value obtained by taking the principal value of $\arg(z)$ and $k = 0$ is called the principal value of $w = \sqrt[n]{z}$.

From the previous results we have that, while in the real field the equation $x^n + a = 0$ may have two, one, or no roots, it has exactly n roots in $\mathbb{C}$. This results may be generalised, as shown in the following Theorem.

Fundamental theorem of algebra

Every polynomial equation of the form

$$a_0 + a_1 z + \cdots + a_n z^n = 0 \quad (a_n \neq 0)$$

with complex number coefficients has exactly n roots (or solutions) in $\mathbb{C}$, if each of them is counted with its multiplicity.

Roots of unity

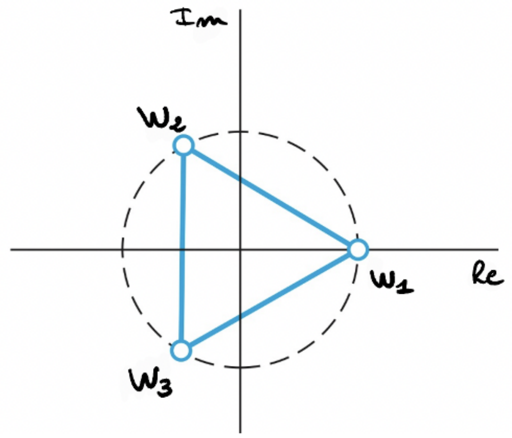
Let us consider $z = 1$ and compute $\sqrt[n]{z}$. We have $r = |z| = 1$ and $\theta = \operatorname{Arg}(z) = 0$. Therefore we have

$$\sqrt[n]{(1)} = \cos\frac{2k\pi}{n} + i\frac{2k\pi}{n}\,, \quad k = 0, 1, \ldots, n-1\,.$$

These n values are called the n-th roots of unity and lie on a circle of circle 1 and center 0 in the complex plane.

For example, for $n = 3$ we get

- $k = 0 \qquad w_1 = 1$
- $k = 1 \qquad w_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $k = 2 \qquad w_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$



Note!

If ω denotes the value of $\sqrt[n]{1}$, then the n values of $\sqrt[n]{1}$ for $k = 0, 1, \ldots, n-1$ can be written as

$$1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1},$$

as can be easily checked by applying De Moivre's formula.

Moreover, from the rule of product of complex numbers in polar form, it can be easily seen that, if w_1 is any n-th root of an arbitrary complex number z $(\neq 0)$, then the n values of $\sqrt[n]{z}$ are

$$w_1,\ w_1\omega,\ w_1\omega^2,\ \ldots, w_1\omega^{n-1}.$$

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