Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4120 Calculus 4K**

Academic contact during examination:

Phone:

Examination date:

Examination time (from–to):

Permitted examination support material: (Code C): Approved simple calculator.

Other information:

- Every answer must be justified; describe clearly how you have reached your answers.

Language: English

Number of pages: 9

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
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sort/hvit ☒ **farger** ☐
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Problem 1 Find the solution $y(t)$ $(t > 0)$ of the following initial value problem by using the Laplace transform:

$$y''(t) + 3y'(t) + 2y = f(t) = \begin{cases} 0, & 0 < t < 1, \\ 1, & 1 < t < 2, \\ 0, & 2 < t, \end{cases} \qquad y(0) = y'(0) = 0.$$

Solution

We use the Laplace transform on the equation to obtain

$$s^2Y(s) + 3sY(s) + 2Y(s) = \mathcal{L}(f)(s)$$
$$= \int_0^\infty e^{-st} f(t)\, dt = \int_1^2 e^{-st}\, dt = \frac{e^{-s} - e^{-2s}}{s}.$$

Solving the resulting algebraic equation gives us

$$(s^2 + 3s + 2)Y(s) = \frac{e^{-s} - e^{-2s}}{s}, \quad Y(s) = \frac{e^{-s} - e^{-2s}}{(s+2)(s+1)s}.$$

Now we use partial fractions to simplify the denominator: Let

$$\frac{1}{(s+2)(s+1)s} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s}.$$

Then

$$A(s+1)s + B(s+2)s + c(s+2)(s+1) = 1$$
$$\iff s^2(A + B + C) + s(A + 2B + 3C) + 2C = 1$$
$$\implies 2C = 1\,,\; A + B + 3C = 0\,,\; A + B + C = 0.$$

From this we see that $C = \frac{1}{2}$, and so $A + B = -\frac{1}{2}$ and $A + 2B = -\frac{3}{2}$. Solving this system of equations, we find that $A = \frac{1}{2}$ and $B = -1$. Thus

$$Y(s) = (e^{-s} - e^{-2s}) \left(\frac{1}{2(s+2)} - \frac{1}{s+1} + \frac{1}{2s} \right).$$

Using the t-shift rule, we have

$$y(t) = \mathcal{L}^{-1}(Y)(t) = u(t-1) \left(\frac{1}{2} e^{-2(t-1)} - e^{-(t-1)} + \frac{1}{2} \right) - u(t-2) \left(\frac{1}{2} e^{-2(t-2)} - e^{-(t-2)} + \frac{1}{2} \right).$$

Problem 2 Let $f(x) = x(1-x)$ for $0 < x < 1$.

a) Find the Fourier cosine series with period 2 which represents $f(x)$ in $0 < x < 1$. Sketch the sum of the series on the interval $-2 < x < 2$.

b) Let $g(x) = \sum_{n=1}^{\infty} n a_n \cos(n\pi x)$, and $h(x) = \sum_{n=1}^{\infty} n^2 a_n \cos(n\pi x)$, where

$$a_n = \frac{-2(1+\cos(n\pi))}{(n\pi)^2}.$$

Check whether the following two integrals are finite:

$$(i) \int_{-1}^{1} |g(x)|^2 \, dx, \qquad (ii) \int_{-1}^{1} |h(x)|^2 \, dx.$$

Hint: Parseval's identity.

c) Find a solution for the following heat equation with Neumann boundary conditions:

$$\begin{aligned} u_t(x,t) &= c^2 u_{xx}(x,t), && x \in [0,1], \quad t > 0, \\ u_x(0,t) &= u_x(1,t) = 0, && t > 0, \\ u(x,0) &= f(x) = x(1-x), \end{aligned}$$

where c is a positive constant. Explain the details. *Hint:* Separation of variables.

Solution

(a) The Fourier cosine series of the function f, with period 2, representing f on **the interval** $(\mathbf{0}, \mathbf{1})$, has the form

$$\sum_{n=0}^{\infty} a_n \cos(n\pi x),$$

where

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\, dx \ , \ n > 0 \ ,$$

and

$$a_0 = \int_0^1 f(x)\, dx.$$

We obtain the coefficients

$$a_0 = \int_0^1 x(1-x)\,dx = \frac{1}{6},$$

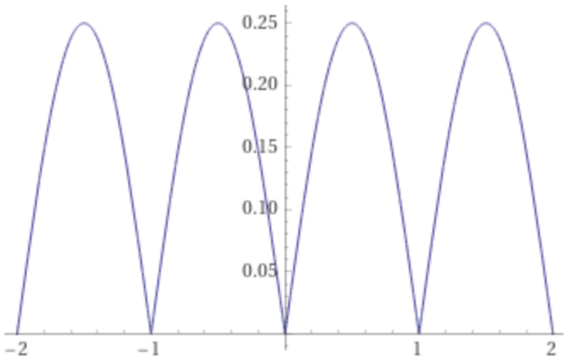
and

$$\begin{aligned} a_n &= 2\int_0^1 x(1-x)\cos(n\pi x)\,dx \\ &= 2\left[\frac{1}{(n\pi)^2}\cos(n\pi x) + \frac{x}{n\pi}\sin(n\pi x)\right]_0^1 \\ &\quad -2\left[\frac{2x}{(n\pi)^2}\cos(n\pi x) - \frac{2-(n\pi x)^2)}{(n\pi)^3} sin(n\pi x)\right]_0^1 \\ &= 2\left(\frac{(-1)^n - 1}{(n\pi)^2} - \frac{2(-1)^n}{(n\pi)^2}\right) = \frac{-2-2(-1)^n}{(n\pi)^2}. \end{aligned}$$

Thus

$$f(x) \sim \frac{1}{6} + \sum_{n=1}^{\infty} \frac{-2-2(-1)^n}{(n\pi)^2}\cos(n\pi x)) \ , \ (0 < x < 1).$$

The graph of the resulting Fourier cosine series is:



(b) • By Parseval's identity, we have

$$\frac{1}{2}\int_{-1}^{1} |g(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2,$$

where

$$c_n = \frac{1}{2}\int_{-1}^{1} g(x)e^{-in\pi x}\,dx = \frac{1}{2}\int_{-1}^{1} g(x)(\cos(n\pi x) - i\sin(n\pi x))\,dx.$$

Since $g(x) = \sum_{n=1}^{\infty} na_n \cos(n\pi x)$,

$$\int_{-1}^{1} \cos(m\pi x)\cos(n\pi x)\, dx = \left\{ \begin{array}{ll} 0, & m \neq n \\ 1, & m = n \neq 0 \end{array} \right\},$$

and

$$\int_{-1}^{1} \cos(m\pi x)\sin(n\pi x)\, dx = 0 \ , \ \forall n, m \in \mathbb{N},$$

we get that $c_n = c_{-n} = \frac{1}{2} na_n$, $n \in \mathbb{N}$ and $c_0 = 0$. Hence

$$\int_{-1}^{1} |g(x)|^2\, dx = 2 \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$= \sum_{n=-\infty, n\neq 0}^{\infty} 2 \left| \frac{(-1-(-1)^n)}{n\pi^2} \right|^2 = \frac{2}{\pi^4} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{((-1)^n + 1)^2}{n^2} < \infty,$$

where the convergence of the series can be also checked by ratio test, etc.

- Similarly, again using Parseval's identity, we have

$$\int_{-1}^{1} |h(x)|^2\, dx = 2 \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{2}{\pi^4} \sum_{n=-\infty, n\neq 0}^{\infty} |-1-(-1)^n|^2 = \infty.$$

(c) We solve this by seperation of variables: Assume a solution u is of the form $u(x,t) = F(x)G(t)$. Then $u_t(x,t) = c^2 u_{xx}(x,t)$ implies that $F(x)G'(t) = c^2 F''(x)G(t)$ for all $x \in [0,1]$ and $t > 0$, and hence

$$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}.$$

As G is a function of t and F is a function of x, we must have

$$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant k. This leads to the equations

$$F'' - kF = 0 \tag{1}$$

and

$$G' - c^2 kG = 0. \tag{2}$$

The boundary conditions gives us that $u_x(0,t) = F'(0)G(t) = F'(1)G(t) = 0$. Since we are seeking nonzero solutions u, we must have $G \neq 0$, which implies that

$$F'(0) = F'(1) = 0. \tag{3}$$

If $k > 0$, then (1) has the solution $F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$ and by (3)

$$F'(0) = \sqrt{k}A - \sqrt{k}B = 0 \implies A = B$$
$$F'(1) = \sqrt{k}Ae^{\sqrt{k}} - B\sqrt{k}e^{-\sqrt{k}} = A\sqrt{k}(e^{\sqrt{k}} - e^{-\sqrt{k}}) = 0$$
$$\implies A = B = 0 \implies F = 0 \implies u = 0,$$

which is of no interest.

If $k = 0$, then (1) gives us that $F(x) = Ax + B$ and (3) implies in turn that $A = 0$, hence $F = B$. By (2), $G(t) = C$ for some constant C, and so $u(x,t) = BC =: D_0$.

If $k < 0$, let $p^2 = -k$. Then (1) has the solution $F(x) = A\cos(px) + B\sin(px)$. Moreover, by (3), $F'(0) = pB = 0$, and thus $B = 0$ (as $p \neq 0$). Also by (3), $F'(1) = -pA\sin(p) = 0$, so that $p = n\pi$ for $n \in \mathbb{N}$, since otherwise if $A = 0$, $F = 0$, which would in turn give us that $u = 0$. Thus $F_n(x) = A_n\cos(n\pi x)$. Now, with $k = -n^2\pi^2$, (2) becomes $G_n'(t) + n^2\pi^2c^2G_n(t) = 0$, which has the solution $G_n(t) = E_ne^{-(n\pi)^2c^2t}$.

We now have, for each $n \in \mathbb{N}$, solutions $u_n(x,t) = D_nF_n(x)G_n(t)$ satisfying both the partial differential equation and the prescribed boundary conditions. By the principle of superposition,

$$u(x,t) = \sum_{n=0}^{\infty} D_nF_n(x)G_n(t) = \sum_{n=0}^{\infty} D_n\cos(n\pi x)\,e^{-(n\pi)^2c^2t},$$

is also such a solution. Now, the initial condition leads to the problem of determining the coefficients D_n such that

$$u(x,0) = \sum_{n=0}^{\infty} D_n\cos(n\pi x) = x(1-x).$$

But from part a) of this problem, we know that these coefficients are precisely of the form

$$D_n = \frac{-2(1+\cos(n\pi))}{(n\pi)^2},$$

and so a solution to our problem is

$$u(x,t) = \sum_{n=0}^{\infty} \frac{-2(1+\cos(n\pi))}{(n\pi)^2}\cos(n\pi x)\,e^{-(n\pi)^2c^2t}.$$

Problem 3

a) For every $z \in \mathbb{C}$ we define $f(z)$ by

$$f(z) = e^{-z^2}.$$

Use the Cauchy–Riemann equations to check if $f(z)$ is analytic in every point $z \in \mathbb{C}$.

b) For every $z \in \mathbb{C}$ we define $g(z)$ by

$$g(z) = h(z) + \frac{1}{(z-1)^2},$$

where

$$h(z) = \begin{cases} e^{-(1/z^2)}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Find and classify all zeros and singularities of $g(z)$ including at infinity. Is $h(z)$ continuous at $z = 0$? Explain.

c) Find all Taylor and Laurent series of $g(z)$ around $z = 0$. State the domains where the series converge.

Solution

(a) We rewrite the function f as follows:

$$\begin{aligned} f(z) = e^{-z^2} &= e^{-(x+iy)^2} = e^{-(x^2-y^2)} e^{-2xyi} \\ &= e^{y^2-x^2}(\cos(2xy) - i\sin(2xy)) = u(x,y) + iv(x,y). \end{aligned}$$

Thus $u(x,y) = e^{-x^2+y^2}\cos(2xy)$ and $v(x,y) = -e^{-x^2+y^2}\sin(2xy)$. We now check if the Cauchy-Riemann equations are satisfied:

$$\begin{aligned} u_x &= -2xe^{-x^2+y^2}\cos(2xy) - 2ye^{-x^2+y^2}\sin(2xy) \\ v_y &= -2ye^{-x^2+y^2}\sin(2xy) - 2xe^{-x^2+y^2}\cos(2xy), \end{aligned}$$

so at least $u_x = v_y$.

$$\begin{aligned} u_y &= 2ye^{-x^2+y^2}\cos(2xy) - 2xe^{-x^2+y^2}\sin(2xy) \\ v_x &= 2xe^{-x^2+y^2}\sin(2xy) - 2ye^{-x^2+y^2}\cos(2xy), \end{aligned}$$

showing that also $u_y = -v_x$. Since the Cauchy-Riemann equations are satisfied everywhere, $f(z) = e^{-z^2}$ is analytic at each $z \in \mathbb{C}$.

(b) *Singularities:* $g(z)$ has an essential isolated singularity at $z = 0$ and a second order pole at $z = 1$ (there is no singularity at infinity). (Note: Since $e^{-(1/z^2)}$ has an essential isolated singularity at $z = 0$, and this is not removable, defining a function value at $z = 0$ does not remove the singularity. See the continuity).

Zeros: $g(z)$ does not have a zero including infinity.

Continuity of $h(z)$ at $z = 0$: For h to be continuous at $z = 0$ we would need to have that $\lim_{z\to 0} h(z) = h(0) = 0$. And this needs to hold regardless of the manner in which z approaches to zero. But, if we let z approach to zero from the positive imaginary axis, i.e. $z = iy$ for $y \geq 0$, we have

$$\lim_{y\to 0} e^{-1/(iy)^2} = \lim_{y\to 0} e^{1/y^2} = \infty.$$

Consequently $h(z)$ is not continuous at $z = 0$.

(c) Recall that

$$e^{-1/z^2} = e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n} n!} \ , \ (z \neq 0).$$

We also have

$$(\frac{1}{1-z})' = \frac{1}{(1-z)^2}.$$

So that

$$\frac{1}{(1-z)^2} = \Big(\sum_{n=0}^{\infty} z^n \Big)' = \sum_{n=1}^{\infty} n z^{n-1} \ , \ (|z| < 1),$$

and

$$\frac{1}{(1-z)^2} = \Big(- \frac{1}{1-1/z} \cdot \frac{1}{z} \Big)' = -\Big(\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} \Big)'$$
$$= \sum_{n=0}^{\infty} (n+1) \frac{1}{z^{n+2}} \ , \ (|z| > 1),$$

where we have used term-wise differentiation. Therefore, we obtain the two Laurent series

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n} n!} + \sum_{n=1}^{\infty} n z^{n-1} \ , \ (0 < |z| < 1),$$

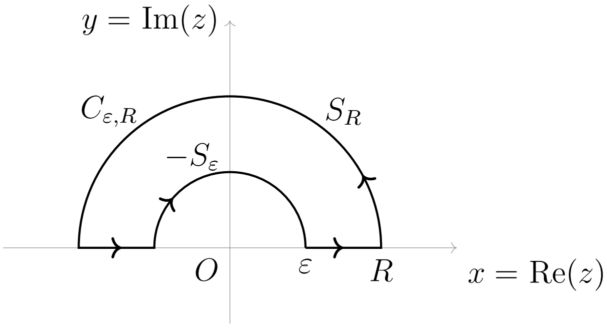
and

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n} n!} + \sum_{n=0}^{\infty} (n+1) \frac{1}{z^{n+2}} \ , \ (|z| > 1).$$

Problem 4 Calculate the following integral

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx.$$

Hint: Integrate $f(z) = e^{iz}/z$ over the closed path $C_{\varepsilon,R}$,



where $S_r : re^{i\theta},\ \theta \in [0, \pi]$ is a half circle of radius $r > 0$, oriented counter clockwise.

You may use without explanation Jordan's Lemma: If $f(z) = e^{iz} g(z)$ for a continuous function $g(z)$ on S_R, then

$$\left| \int_{S_R} f(z) dz \right| \leq \pi \max_{z \in S_R} |g(z)|.$$

Solution Since $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$,

$$I := \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \text{P.V.} \frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{iz} - e^{-iz}}{z} dz = \text{P.V.} \frac{1}{i} \int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz,$$

where the last equality is due to the fact that

$$\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz = \int_{-\infty}^{\infty} \frac{-e^{-iz}}{z} dz.$$

Now,

$$\frac{1}{i} \lim_{\substack{\epsilon \to 0^+ \\ R \to \infty}} \oint_{C_{\epsilon,R}} \frac{e^{iz}}{z}\, dz = I + \frac{1}{i} \lim_{\epsilon \to 0^+} \int_{-S_\epsilon} \frac{e^{iz}}{z}\, dz + \frac{1}{i} \lim_{R \to \infty} \int_{S_R} \frac{e^{iz}}{z}\, dz.$$

Let us write this as $A = I + B + C$. The function $\frac{e^{iz}}{z}$ has singularities only at $z = 0$, which for each $R > \epsilon > 0$ lies outside of the contour $C_{\epsilon,R}$. Hence by Cauchy's integral theorem, we have that $A = 0$. To calculate B, we note that $\frac{e^{iz}}{z}$ has an order 1 pole at $z = 0$, and so

$$B = \frac{1}{i} \lim_{\epsilon \to o^+} \int_{-S_R} \frac{e^{iz}}{z}\, dz = \frac{-1}{i} \pi i \operatorname{Res}_{z=0}\Big[e^{iz}/z\Big] = -\pi.$$

To calculate C, we use Jordan's Lemma:

$$\Big| \int_{S_R} \frac{e^{iz}}{z}\, dz \Big| \leq \pi \max_{z \in S_R} \Big| e^{iz}/z \Big| = \pi \max_{z \in S_R} \Big| 1/z \Big| = \frac{\pi}{R} \to 0, \text{ as } R \to \infty.$$

Thus $C = 0$, and we conclude that

$$I = A - B - C = 0 - (-\pi) - 0 = \pi.$$