TMA 4120 Exam. 17. VIII. 2021

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① $$y(t) + 3\int_0^t y(\tau)\,d\tau = \delta(t-5)$$

$$Y(s) + 3\cdot\frac{1}{s}Y(s) = e^{-5s}$$

$$Y(s) = \frac{1}{1+\frac{3}{s}}e^{-5s} = \frac{s}{s+3}e^{-5s}$$

$$= e^{-5s} - \frac{3}{s+3}e^{-5s}$$

$$y(t) = \delta(t-5) - 3e^{-3(t-5)}u(t-5)$$

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② $$\sum_{n=1}^{\infty} b_n \cdot 1 \cdot \sin(nx) \overset{2}{\doteq} 4(\sin(x))^3, \quad 0 \le x \le \pi$$

Notice that $4(\sin(x))^3 = 3\sin(x) - \sin(3x)$.

This follows, for example, from

$$4\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 = \cdots\cdots$$

Thus

$$u(x,t) = -e^{-9t}\sin(3x) + 3e^{-t}\sin(x).$$

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⑤ Answer: $v(x,0) = \dfrac{1}{1+x^2}$

according to the theory of the Heat Equation.

(3) The function is analytic except at the two points where

$$z^2 + z + 1 = 0 \iff z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The radius of convergence is equal to the distance from the center $z=0$ to the nearest singularity, i.e.

$$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$$

(4a) $$g(z) = \frac{1+e^{i\pi z}}{z(1+z)^2}$$

The poles are the points $z=0$ and $z=-1$.

<u>$z=0$</u> This is a simple pole since

$$\lim_{z\to 0} z g(z) = 2 \quad (\neq 0, \neq \infty)$$

<u>$z=-1$</u> The order of the pole is <u>at most</u> 2. In fact, it is of order one (= simple) because

$$1+e^{i\pi z} = 1 + e^{i\pi(z+1)} e^{i\pi} = 1 + e^{i\pi}\left[1 + i\pi(z+1) + \cdots\right]$$

$$= -i\pi(z+1) + O(z+1)^2 \leftarrow \text{Higher terms.}$$

so that one factor $z+1$ is cancelled.

(4b) $$\oint g(z)\,dz = 2\pi i \left\{ \operatorname*{Res}_{z=0}\{g(z)\} + \operatorname*{Res}_{z=-1}\{g(z)\} \right\}$$

$$\operatorname*{Res}_{z=0}\{g(z)\} = 2 \quad (\text{see } 4a)$$

$$g(z) = \frac{-i\pi + \frac{1}{2}\pi^2 (z+1) + \text{higher terms}}{z(z+1)}$$

Since the pole at $z=-1$ is simple,

$$\operatorname*{Res}_{z=1}\{g(z)\} = \lim_{z\to -1} (z+1) g(z) = \frac{-i\pi + 0}{-1} = i\pi$$

It follows that

$$\oint g(z)\,dz = 2\pi i \{ 2 + i\pi \}.$$

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(6) By direct differentiation

$$\Delta(uv) = u\Delta v + v\Delta u + 2\nabla u \cdot \nabla v$$

The functions are harmonic and hence $\Delta u = 0$, $\Delta v = 0$. We have

$$\Delta(uv) = 2\nabla u \cdot \nabla v$$

The Cauchy-Riemann equations imply that

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0.$$

<u>Answer:</u> $\Delta(uv) = 0.$