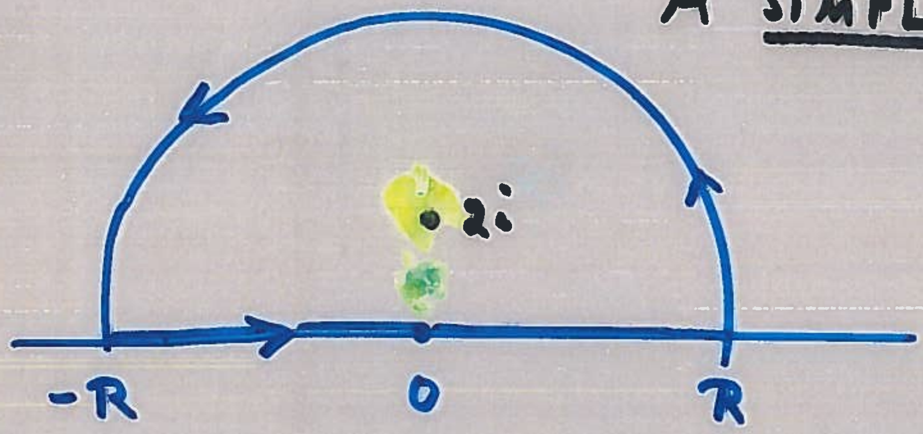
¿ $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,dx = \frac{\pi}{2e^6}$?

USE:

$$f(z) = \frac{e^{3iz}}{z^2+4} \qquad \cancel{\frac{\cos 3z}{z^2+4}} \quad \text{NO!}$$

POLES: $\pm 2i$ Only $+2i$ is in the upper half plane.

A **SIMPLE** POLE.



By the residue theorem

$$2\pi i \cdot \operatorname*{Res}_{z=2i} f(z) = \oint f(z)\,dz = \int_{-R}^{R} f(x)\,dx$$

Diameter

$$+ \int f(z)\,dz$$

Half-circle

The integral along the arc approaches zero as $R \to \infty$:

$$|f(z)| = \frac{|e^{3i(x+iy)}|}{|z^2+4|} = \frac{e^{-3y}}{|z^2+4|}$$

$$\leq \frac{1}{|z^2|-4} = \frac{1}{R^2-4}$$

$$\left| \int_{\Gamma} f(z)\,dz \right| \underset{\text{ML-Ineq}}{\leq} \frac{\pi R}{R^2-4} \xrightarrow[R\to\infty]{} 0 .$$

THUS

$$2\pi i \cdot \operatorname*{Res}_{z=2i} f(z) = \int_{-\infty}^{\infty} f(x)\,dx + 0 .$$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i} \frac{(z-2i)e^{3iz}}{\underbrace{z^2+4}_{(z-2i)(z+2i)}} = \frac{e^{-6}}{4i}$$

RESULT:

$$\int_{-\infty}^{\infty} \frac{e^{3ix}\,dx}{x^2+4} = \frac{\pi}{2e^6}.$$

$$e^{3ix} = \cos 3x + i \sin 3x,$$

$$\int_{-\infty}^{+\infty} \frac{\cos 3x}{x^2+4}\,dx + i\int_{-\infty}^{\infty} \frac{\sin 3x}{x^2+4}\,dx = \frac{\pi}{2e^6},$$

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,dx = \frac{\pi}{2e^6}, \\ \displaystyle\int_{-\infty}^{\infty} \frac{\sin 3x}{x^2+4}\,dx = 0 \end{cases}.$$