Problems about complex numbers

one or two problems each week

(1) Let $z = x + iy$. Find the real and imaginary parts of the following numbers
  a) z^2;
  b) $\frac{1}{z}$;
  c) $\bar{z}$;

(2) Let $z = re^{i\varphi}$. Find the absolute values and arguments of the following numbers
  a) z^2,
  b) $\frac{1}{z}$;
  c) $\bar{z}$.

(3) Find the absolute values and arguments of the following complex numbers:
  a) $1 + i^{815}$;
  b) $(1+i)(1-i)^{-1}$;
  c) $-\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$.

(4) Make pictures of the sets of complex numbers, which are described by the following relations
  a) $\mathrm{Re} z > -1$;
  b) $|\mathrm{Im} z - 1| < 3$;
  c) $|z - i| < 1$;
  d) $0 < \arg z < \frac{\pi}{3}$;
  e) $\mathrm{Re}(ze^{i\pi/4}) > 0$.

(5) Which lines in the complex plane are described by the relations
  a) $\mathrm{Im}(ze^{-i\pi/3}) = 1$;
  b) $\mathrm{Re}\frac{1}{z} = 4$;
  c) $|z - i| = |z - 1|$.

(6) Prove the relation

$$\frac{1}{2} + \cos\varphi + \cos 2\varphi + \ \dots \ + \cos n\varphi = \frac{\sin\left(\frac{n+1}{2}\right)\varphi}{2\sin\frac{\varphi}{2}}$$

Hint. Consider the geometric progression $1 + e^{i\varphi} + \ \dots \ + e^{in\varphi}$.