EXAMPLE (WITH INTERFERENCE)

$$\boxed{\frac{d^2y}{dt^2} + 0{,}02\,\frac{dy}{dt} + 25\,y = r(t)}$$

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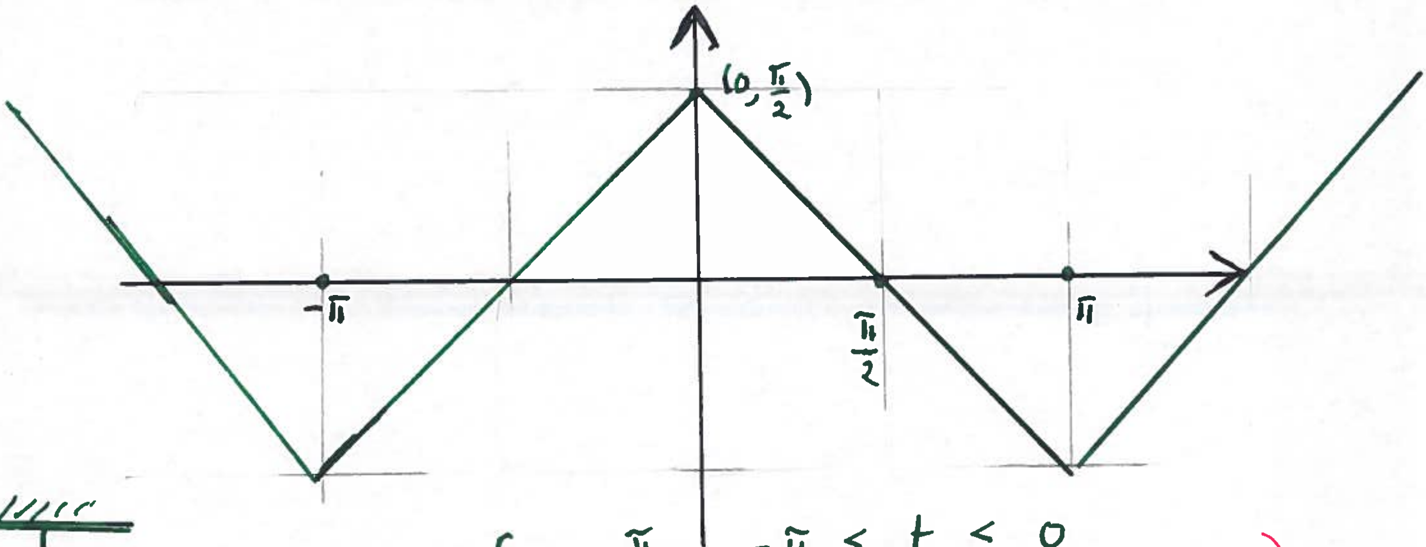
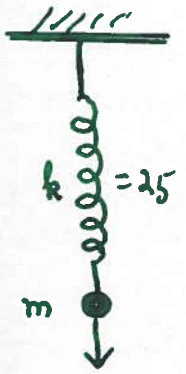




$$r(t) = \begin{cases} t + \frac{\pi}{2}, & -\pi \le t \le 0 \\ -t + \frac{\pi}{2}, & 0 \le t \le \pi \end{cases}$$

(given)

$$r(t) \equiv r(t + 2\pi) \qquad \underline{\text{period}} = 2\pi$$

Notice that the *free* mass-spring system, i.e. $r(t) \equiv 0$, has the general solution

OBS! $\approx 5t$ EXACT

$$y(t) = e^{-\frac{t}{100}} \underbrace{\left(A\cos(4{,}99999\,t) + B\sin(4{,}99999\,t)\right)}_{= C\cos(4{,}99999\,t - \delta)},$$

which is easy to find. As we shall see, the dominating term in the general solution (with $r(t)$ as above) is

$$0{,}5092958\ldots \sin(5t).$$

(1) First, we expand the forcing term as a Fourier series:

$$r(t) = \frac{4}{\pi}\left(\cos(t) + \frac{1}{3^2}\cos(3t) + \frac{1}{5^2}\cos(5t) + \cdots\right)$$

(2) Then we solve (only one term from $r(t)$!)

$$y'' + 0{,}02\,y' + 25\,y = \frac{4}{\pi n^2}\cos(nt), \quad n = \text{odd}$$

using the ANSATZ:

$$\begin{cases} y = A_n\cos(nt) + B_n\sin(nt) \\ y' = -nA_n\sin(nt) + nB_n\cos(nt) \\ y'' = -n^2 y \end{cases}$$

Inserting, we have

$$[25-n^2][A_n\cos(nt) + B_n\sin(nt)] + 0{,}02n[-A_n\sin(nt) + B_n\cos(nt)] = \frac{4}{\pi n^2}\cos(nt).$$

The terms cancel if

$$\begin{cases} (25-n^2)A_n + 0{,}02\,n B_n = \dfrac{4}{\pi n^2} \\ -0{,}02n\,A_n + (25-n^2)B_n = 0 \end{cases}$$

linear system unknown: A_n, B_n

Hence, writing $D_n = (25-n^2)^2 + (0{,}02n)^2$,

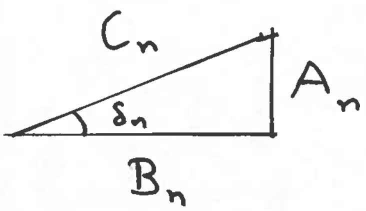
$$A_n = \frac{4(25-n^2)}{n^2\pi D_n}, \quad B_n = \frac{0{,}08}{n\pi D_n}$$

$n = 1, 3, 5, 7, 9, \ldots$

③ Superposition yields the final solution

$$y(t) = e^{-t/100}\left[A\cos(4{,}99999t) + B\sin(4{,}99999t)\right]$$

$$+ \sum_{n=1}^{\infty} \underbrace{A_n\cos(nt) + B_n\sin(nt)}_{C_n \sin(nt - \delta_n)}$$



$$C_n = \sqrt{A_n^2 + B_n^2} \quad (\hat{=} \text{ amplitude})$$

$$(C_{2n} = 0)$$

$C_1 \approx 0{,}0530$

$C_3 \approx 0{,}0088$

$C_5 \approx 0{,}5093$! This dominates.

$C_7 \approx 0{,}0011$

$C_9 \approx 0{,}0003$

$C_{11} \approx 0{,}0001$

$\vdots$

The interference comes from

$5t \approx 4{,}99999t$

(The term $n = 5$ is

$0{,}5092558\ldots \sin(5t)$.)

$A_5 = 0$, $B_5 = 8/5\pi$