

$$\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}} \quad (a > 0)$$

We calculate this, for simplicity, when  $a = 1/2$ .

$$F(\omega) \stackrel{\text{DEF.}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-i\omega x} dx$$

DIFFERENTIATE  
 $\frac{d}{d\omega}$

$$F'(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{(-x e^{-\frac{x^2}{2}})}_{= \frac{d}{dx} e^{-x^2/2}} i e^{-i\omega x} dx \quad [\text{INTEGRATION BY PARTS}]$$

$\frac{d}{d\omega}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} i e^{-i\omega x} dx + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-i\omega x} dx$$

$|i e^{-i\omega x}| = 1$   
 $= 0$

! Here  $F(\omega)$  appears again!

$$= -\omega F(\omega) \quad !$$

Thus

$$\boxed{\frac{dF}{d\omega} + \omega F = 0} \Leftrightarrow F(\omega) = C e^{-\frac{\omega^2}{2}}$$

DIFF. EQN.

In order to determine the integration constant, we set  $\omega = 0$

$$C \cdot 1 = F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = 1$$

It follows that

$$\widehat{e^{-\frac{x^2}{2}}} = e^{-\frac{\omega^2}{2}}$$

Important integral.