# Forelesning 4 Laplace

torsdag 13. august 2020 13:09

Recall: Last time we solved ODE of the form

$$y'' + ay' + b = r(t), \qquad y(0) = K_0, \quad y'(0) = K_1$$

Taking the Laplace transform, we obtain the subsidiary problem

$$Y(s^2 + as + b) = K_0(a+s) + K_1 + R(s)$$

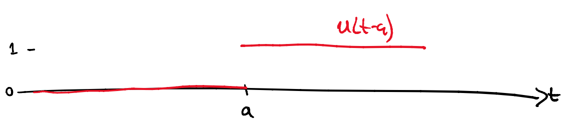
$$Y = \underbrace{\frac{1}{s^2+as+b}}_{Q(s),\ \text{transfer function.}} \left(K_0(a+s) + K_1 + R(s)\right)$$

$$\Rightarrow y = \mathcal{L}^{-1}\{Y\}$$

Unit step function    Heaviside function

The unit step function is defined as

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$




Who cares?
It is very comon in solving ODE's within electrical engineering.

Ex    Let $i(t)$ denotes current.

$$i' + bi = \underbrace{\begin{cases} 0 & t < 5 \\ 10 & t > 5 \end{cases}}_{10\,u(t-5)}$$

What is $\mathcal{L}\{u(t-a)\}$?

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\,dt$$

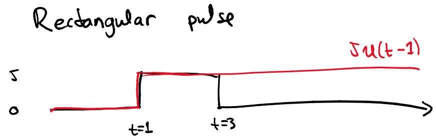
$$= \int_0^a \underset{\overset{\parallel}{0}}{u(t-a)}\, e^{-st}\,dt + \int_a^\infty \underset{\overset{\parallel}{1}}{u(t-a)}\, e^{-st}\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{-1}{s} e^{-st}\right]_a^\infty$$

$$= \underline{\underline{\frac{1}{s} e^{-as}}}$$

Ex    Rectangular pulse




$$\boxed{f(t) = 5u(t-1) - 5u(t-3)}$$

$$f(t<1) = 5\cdot 0 \quad - 5\cdot 0 \quad = 0$$

$$f(1<t<3) = 5\cdot 1 \quad - 5\cdot 0 \quad = 5$$

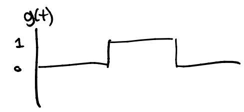
$$f(t>3) = 5\cdot 1 \quad - 5\cdot 1 \quad = 0$$

$$\mathcal{L}\{f(t)\} = 5\mathcal{L}\{u(t-1)\} - 5\mathcal{L}\{u(t-3)\}$$
$$= 5\cdot\frac{1}{s}e^{-1s} - 5\cdot\frac{1}{s}e^{-3s}$$
$$= \frac{5}{s}\left(e^{-s} - e^{-3s}\right)$$

<u>Ex</u>

$$f(t) = \begin{cases} 0 & t<2\pi \\ \sin(t) & 2\pi<t<4\pi \\ 0 & t>4\pi \end{cases}$$


$$g(t) = \begin{cases} 0 & t<2\pi \\ 1 & 2\pi<t<4\pi \\ 0 & t>4\pi \end{cases}$$




$$g(t) = u(t-2\pi) - u(t-4\pi)$$

$$\Rightarrow \quad f(t) = \sin(t)\,g(t)$$
$$= \sin(t)\left(u(t-2\pi) - u(t-4\pi)\right)$$

## <u>t-shifting</u>

<u>Theorem</u> We have that

$$\boxed{\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t)\}}$$

If $F(s) = \mathcal{L}\{f(t)\}$, this is equivalent to

$$\boxed{f(t-a)\,u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}}$$

<u>Proof</u>

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = \int_0^\infty f(t-a)\underbrace{u(t-a)}_{\substack{=0 \text{ for } t<a \\ =1 \text{ for } t>0}} e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)\,e^{-st}\,dt$$

Substitution: $\tau = t-a$, $t = \tau + a$, $dt = d\tau$

$$= \int_0^\infty f(\tau)\,e^{-s(\tau+a)}\,d\tau$$

$$= e^{-sa}\underbrace{\int_0^\infty f(\tau)\,e^{-s\tau}\,d\tau}_{\mathcal{L}\{f(t)\}}$$

$$= e^{-sa}\,\mathcal{L}\{f(t)\} \qquad \blacksquare$$

<u>Ex</u>

$$f(t) = \begin{cases} 0 & t<2\pi \\ \sin(t) & 2\pi<t<4\pi \\ 0 & t>4\pi \end{cases}$$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$$= \sin(t)\left(u(t-2\pi) - u(t-4\pi)\right)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\underbrace{\sin(t)}_{2\pi\text{-periodic}}\,u(t-2\pi)\} - \mathcal{L}\{\sin(t)\,u(t-4\pi)\}$$

$\Rightarrow \sin(t) = \sin(t-2\pi) = \sin(t-4\pi)$

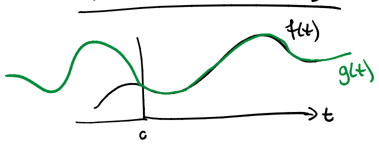
$$= \mathcal{L}\{\sin(t-2\pi)\,u(t-2\pi)\}_{a=2\pi} - \mathcal{L}\{\sin(t-4\pi)\,u(t-4\pi)\}_{a=4\pi}$$

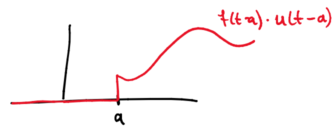
$$= e^{-2\pi s}\,\mathcal{L}\{\sin(t)\} - e^{-4\pi s}\,\mathcal{L}\{\sin(t)\}$$

$$= e^{-2\pi s}\cdot\frac{1}{s^2+1} - e^{-4\pi s}\cdot\frac{1}{s^2+1}$$

$$= \frac{1}{s^2+1}\left(e^{-2\pi s} - e^{-4\pi s}\right)$$

## What is t-shifting




$\mathcal{L}\{f\} = \int_0^\infty f(t)\, e^{-st}\, dt$

$= \int_0^\infty g(t)\, e^{-st}\, dt$

$= \mathcal{L}\{g\}$

$\mathcal{L}\{f(t-a)\}$

**Ex** $\quad \mathcal{L}^{-1}\left\{ \frac{e^{-3s}}{(s+2)^2} \right\} = \mathcal{L}^{-1}\left\{ \underline{e^{-3s}} \cdot \underline{\frac{1}{(s+2)^2}} \right\}$

↳ indicates s-shifting

↳ indicates t-shifting

$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$?

$F(s) := \frac{1}{s^2}$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$

s-shifting theorem says that $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$

choosing $a=2$ gives

$\frac{1}{(s+2)^2} = F(s+2) = \mathcal{L}\{e^{-2t} f(t)\}$

$= \mathcal{L}\{e^{-2t}\, t\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}\, t$

$\mathcal{L}^{-1}\{e^{-as}\, G(s)\} = g(t-a)\, u(t-a)$

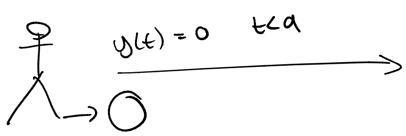
$G(s) = F(s+a) = \frac{1}{(s+2)^2}$

$g(t) = e^{-2t}\, t$

$\mathcal{L}^{-1}\left\{ e^{-3a} \frac{1}{(s+2)^2} \right\} = e^{-2(t-3)}(t-3)\, u(t-3)$

$= \begin{cases} 0 & t<3 \\ e^{-2(t-3)}(t-3) & t>3 \end{cases}$

## Short impulses




At time $t=a$, the football player kicks the ball.
Right after the kick, the ball has velocity 1 (m/s)

$t<a$: $\quad y(t)=0$

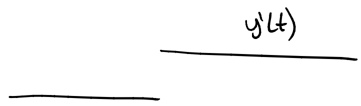
$y'(t)=0$

$y''(t)=0$

Right after $t=a$: $\quad y'(t)=1$

We have that $\quad y'(t) = u(t-a)$

What is $y''(t=a)$ ?  $y''(t<a)=0$
$y''(t>a)=0$

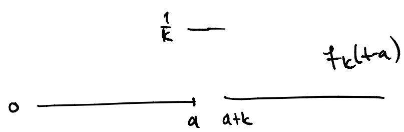



How do we model this?

Assume that the force is applied aver a small time frame $[a, a+k]$

$y'(a+k)=1$
$y'(a)=0$

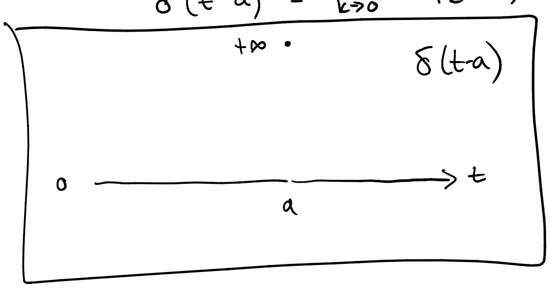
$$y''(t) = f_k(t-a) = \begin{cases} 0 & t<a \\ \frac{1}{k} & a<t<t+k \\ 0 & t>a+k \end{cases}$$




$$\begin{aligned} y'(a+k) &= \int_0^{a+k} y''(t)\,dt \\ &= \int_0^{a+k} f_k(t-a)\,dt \\ &= \int_a^{a+k} \frac{1}{k}\,dt \\ &= \left[\frac{t}{k}\right]_a^{a+k} \\ &= \frac{\cancel{a}+k}{k} - \frac{\cancel{a}}{k} \\ &= 1 \end{aligned}$$

Define The Dirac delta function

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a)$$



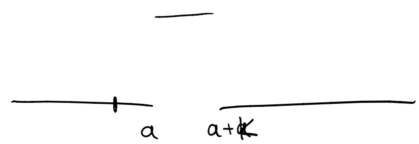

$$\delta(t-a) = \begin{cases} +\infty & t=a \\ 0 & t\neq 0 \end{cases}$$

$$y'' = \delta(t-a)$$

$y'(t) = \int^t y''(\tau)\,d\tau$

$$y'' = \delta(t-a)$$

$$\begin{aligned}
y'(t) &= \int_0^t y''(\tau)\,d\tau \\
&= \int_0^t \delta(\tau-a)\,d\tau \\
&= \int_0^t \lim_{k\to 0} f_k(\tau-a)\,d\tau \\
&= \lim_{k\to 0} \int_0^t f_k(\tau-a)\,d\tau \\
&= \lim_{k\to 0} \begin{cases} 1 & t>a+k \\ ? & \\ 0 & t<a \end{cases} \\
&= \begin{cases} 1 & t>a \\ 0 & t<a \end{cases} \\
&= u(t-a)
\end{aligned}$$




"Property"

$$\begin{aligned}
f(t)\,\delta(t-a) &= f(t)\begin{cases} \delta(0) & t=a \\ 0 & t\neq a \end{cases} \\
&= \begin{cases} f(t)\,\delta(0) & t=a \\ f(t)\cdot 0 & t\neq a \end{cases} \\
&= \begin{cases} f(a)\,\delta(0) & t=a \\ f(a)\cdot 0 & t\neq a \end{cases} \\
&= f(a)\begin{cases} \delta(0) & t=a \\ 0 & t\neq a \end{cases}
\end{aligned}$$

$$\boxed{f(t)\,\delta(t-a) = f(a)\,\delta(t-a)}$$

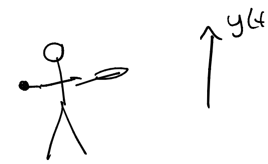
The actual property

$$\boxed{\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)}$$

$$\begin{aligned}
\mathcal{L}\{\delta(t-a)\} &= \int_0^\infty \delta(t-a)\,e^{-st}\,dt \\
&= e^{-sa}
\end{aligned}$$

$$\frac{1}{s}e^{-sa} = \mathcal{L}\{u(t-a)\}$$
$$= \mathcal{L}\left\{\int_0^t \delta(\tau-a)\,d\tau\right\}$$
$$= \frac{1}{s}\mathcal{L}\{\delta(t-a)\}$$
$$= \frac{1}{s}e^{-sa}$$

Ex




Let $y(t)$ be the vertical position of a ball

$y(t) = 0$ for $t < 5$

At tim $t=5$, the ball is hit from below and starts flying upwards with speed $y' = 1$.

$$y'' = \delta(t-5) + \begin{cases} 0 & t<5 \\ -9.81 & t>5 \end{cases}$$

$$\mathcal{L}\{\delta(t-5)\} = \int_0^\infty \delta(t-5)e^{-st}\,dt = e^{-5s}$$

$$= \delta(t-5) - 9.81\,u(t-5)$$

$$s^2Y - s\underset{0}{\underset{\parallel}{y(0)}} - \underset{0}{\underset{\parallel}{y'(0)}} = \mathcal{L}\{\delta(t-5)\} - 9.81\,\mathcal{L}\{u(t-5)\}$$

$$s^2Y = e^{-5s} - 9.81\cdot\frac{1}{s}e^{-5s}$$

$$Y = \frac{1}{s^2}e^{-5s} - \frac{9.81}{s^3}e^{-5s}$$
$$= e^{-5s}\left(\frac{1}{s^2} - 9.81\frac{1}{s^3}\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{e^{-5s}\left(\frac{1}{s^2} - 9.81\frac{1}{s^3}\right)\right\}$$

$$F(s) = \frac{1}{s^2} - 9.81\frac{1}{s^3}$$
$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 9.81\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$
$$= t - 9.81\cdot\frac{1}{2}t^2$$

$$= \mathcal{L}^{-1}\{e^{-5s}F(s)\}$$
$$= f(t-5)\,u(t-5)$$

$$= f(t-5)\,u(t-5)$$

$$= \left(t-5 - \frac{9.81}{2}(t-5)^2\right)u(t-5)$$

$$= \begin{cases} t-5 - \frac{9.81}{2}(t-5)^2 & t>5 \\ 0 & t<0 \end{cases}$$