EXAM in TMA4120, Matematikk 4K

Problem 1 Solve the equation

$$y(t) + 3\int_0^t y(\tau)\,d\tau \;=\; \delta(t-5)$$

using the Laplace transform.

Problem 2 We know that the solutions $u = u(x,t)$ of the problem

$$\begin{cases} \frac{\partial u}{\partial t} \;=\; \frac{\partial^2 u}{\partial x^2}, & t \geq 0,\ 0 \leq x \leq \pi \\ u(0,t) \;=\; u(\pi,t) \;=\; 0, & t \geq 0 \end{cases}$$

are given by the formula

$$u(x,t) \;=\; \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$

Assume that $u(x,t)$, in addition, has the initial values

$$u(x,0) \;=\; 4\big(\sin(x)\big)^3, \qquad 0 \leq x \leq \pi.$$

Find $u(x,t)$! *Hint:* e^{ix}.

Problem 3 Find the radius of convergence for the power series

$$\frac{1}{z^2+z+1} \;=\; 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Problem 4a Find the poles of the function

$$g(z) \;=\; \frac{1+e^{i\pi z}}{z(z+1)^2}$$

and determine their order.

Problem 4b Calculate the integral

$$\oint g(z)\, dz \ = ?$$

along the circle $|z+1| = 5$.

Problem 5 Let

$$v(x,t) \ = \ \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \frac{\exp(-\frac{|x-y|^2}{4t})}{1+y^2}\, dy.$$

Determine

$$\lim_{t\to 0+} v(x,t) \ = ?$$

Hint: Recall a partial differential equation.

Problem 6 Let

$$\Delta \ = \ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

denote the Laplace operator. Suppose that $f(z) = u(x,y) + i\, v(x,y)$ is an analytic function. Is it true that

$$\Delta(uv) \ = \ 0 \ ?$$

(A proof or a counterexample!)