IV COMPLEX ANALYSIS

$$f(z) = u(x,y) + i\,v(x,y) \qquad i^2 = -1$$

TMA 4175

$$z = x + iy \qquad [\text{Ch } 13-16]$$

LAPLACE TRANSFORM

- Used to solve initial value problems for differential eqs.
- A function $f(t)$ becomes a function $F(s)$

DEF.:

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt}$$

LAPLACE TR. OF $f(t)$

<u>Ex.:</u> $\mathcal{L}\{t\} = ?$. Here $f(t) = t$.

$$F(s) = \int_0^\infty e^{-st} \cdot t \, dt$$

$$= \int_0^\infty t \frac{d}{dt}\left(\frac{e^{-st}}{-s}\right) dt$$

$$= \underbrace{\left. t \cdot \frac{e^{-st}}{-s} \right|_0^\infty}_{= 0 \text{ if } \underline{s > 0}} + \underbrace{\frac{1}{s} \int_0^\infty e^{-st} \, dt}_{+ 1/s^2}$$

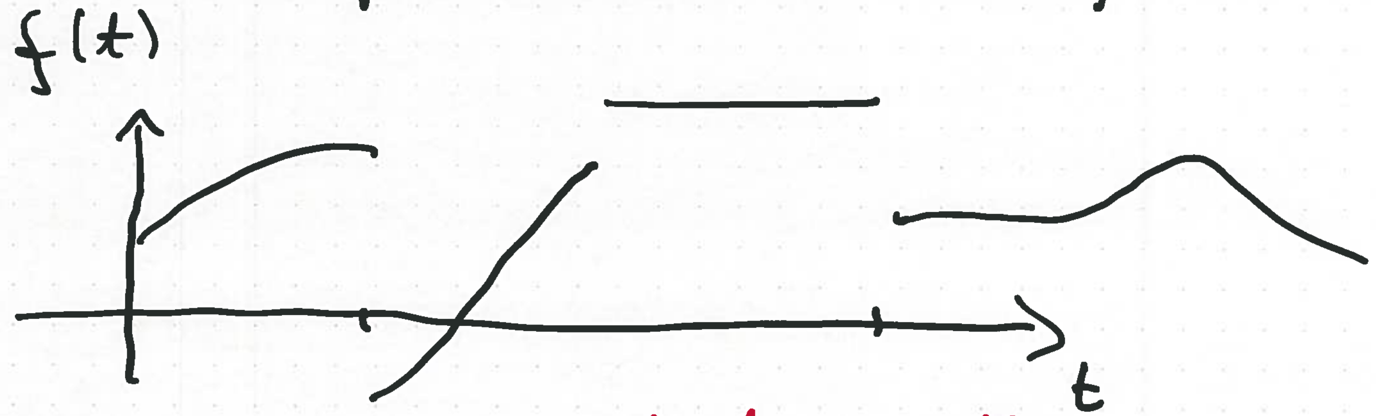
<u>Answer:</u> $\mathcal{L}\{t\} = \frac{1}{s^2} \quad (s > 0)$

COMMENTS

1°) $f(t)$ has to be defined for $t > 0$.

Often $f(0+) = \lim_{t \to 0+} f(t)$

2°) We use piecewise continuous functions



3°) At most exponential growth

$$|f(t)| \leq M e^{+\gamma t}$$

4°) Usually $F(s)$ becomes defined only when $s > \gamma$ (some number)

EX $\mathcal{L}\{e^{at}\} = ?$. Now $f(t) = e^{at}$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty \underbrace{e^{-st} e^{at}}_{e^{t(a-s)}} \, dt$$

$$\boxed{e^{x+y} = e^x e^y}$$

$$= \left. \frac{e^{t(a-s)}}{a-s} \right|_0^\infty = \frac{1}{s-a} \quad \text{if } a-s<0$$

Answer: $\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$, if $s>a$

LINEARITY / SUPERPOSITION

$$\boxed{\mathcal{L}\{a f(t) + b g(t)\} = a F(s) + b G(s)}$$

Proof: The transform is an integral!

Ex $\mathcal{L}\{\cosh(at)\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}\underline{\underline{\mathcal{L}\{e^{at}\}}} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left(\underline{\underline{\frac{1}{s-a}}} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

when $s > |a|$.

s - SHIFTING

$$\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$$

Proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$\mathcal{L}\{\underbrace{e^{at} f(t)}\} = \int_0^\infty e^{-st} \cdot \underbrace{e^{at} f(t)}\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$\boxed{e^{x+y} = e^x e^y}$$

$$= F(s-a) \qquad \square$$

Ex.: $\mathcal{L}\{e^{3t}\cos(\omega t)\} = ?$

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} = \text{"}F(s)\text{"}$$

$$\mathcal{L}\{e^{3t}\cos(\omega t)\} = \frac{s-\omega}{(s-3)^2+\omega^2}$$

RULES FOR DERIVATIVES

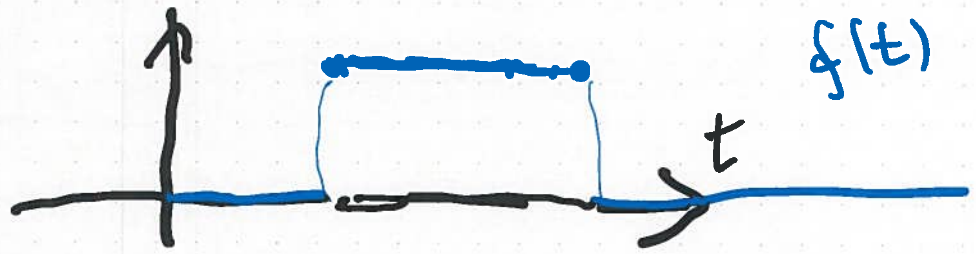
$$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$$

1°) $f(t)$ continuous, when $t \geq 0$.

Important: $f(0) = \lim_{t \to 0+} f(t)$.

2°) $|f(t)| \leq M e^{\gamma t}$

3°) $f'(t)$ piecewise continuous.



$$\boxed{\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0)}$$

<u>Ex.:</u> $\begin{cases} y''(t) - y(t) = t, & t > 0 \\ y(0) = 1, \ y'(0) = 1 \end{cases}$

$$\mathcal{L}\{y''\} - \underline{\mathcal{L}\{y\}} = \mathcal{L}\{t\}$$ superpos.

$$s^2 Y(s) - s \cdot 1 - 1 - \underline{Y(s)} = \frac{1}{s^2}$$

$$(s^2-1)Y(s) = s+1+\frac{1}{s^2}$$

$$Y(s) = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

$$= \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

PROBLEM $y = ?$

$$y(t) = e^{t} + \sinh(t) - t$$
$$= \frac{3}{2}e^{t} - \frac{1}{2}e^{-t} - t$$

Explanation, assuming f' is cont.

$$\mathcal{L}\{f'(t)\} \overset{\text{DEF.}}{=} \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \underbrace{\Big/_0^\infty e^{-st} f(t)}_{0 - e^{0} f(0)} + \underbrace{\int_0^\infty s\, e^{-st} f(t)\,dt}_{s\,\mathcal{L}\{f(t)\}}$$

$\Leftarrow |f(t)| \leq Me^{pt}$

Thus we got $\mathcal{L}\{f'\} = sF(s) - f(0)$, as desired.

PROCEDURE TO SOLVE

$$\begin{cases} a y''(t) + b y'(t) + c y(t) = f(t) \\ y(0) = y_0, \; y'(0) = y_0' \end{cases} \qquad \underline{t > 0}$$

- Take the $\mathcal{L}$-transform of "everything in sight."
- Use the formulas for $\mathcal{L}\{y'\}$ and $\mathcal{L}\{y''\}$ [Only $Y(s)$ survives!]
- Now, find $Y(s)$.
- Finally $y(t)$ #