$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

$$|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x$$

JACOBIAN

$$= u_x^2 + v_x^2 = u_y^2 + v_y^2$$

CAUCHY-R.

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2}$$

<u>Consequence of C-R.</u>

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial y \partial x} \\[2ex] \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y} \end{cases} \Longrightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The functions u and v satisfy the LAPLACE eqn.!

<u>Necessary</u>: $\nabla^2 u = 0$,
$\nabla^2 v = 0$.

<u>Ex:</u> $f(z) = e^x + iy$ is NOT analytic, since $\nabla^2 e^x = e^x \neq 0$.

<u>How to construct the conjugate function</u>

<u>Ex.</u> $f(z) = 2x(1-y) + i\,v(x,y)$ $\qquad v(x,y) \stackrel{?}{=}$

1°) $\nabla^2[2x(1-y)] = \dots = 0$ Hence $u(x,y)$ is harmonic. Thus the problem is possible.

2°) $\dfrac{\partial u}{\partial x} = 2(1-y) = \dfrac{\partial v}{\partial y} \Longleftrightarrow v = 2y - y^2 + \underline{\underline{\underline{C(x)}}}$!

Function

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial y \partial x} \\[2ex] \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y} \end{cases} \Longrightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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How to construct the conjugate function

Ex. $f(z) = 2x(1-y) + i\,v(x,y)$ OUR PROBLEM: $\underline{v(x,y) = ?}$

1°) $\nabla^2[2x(1-y)] = \dots = 0$ Hence $u(x,y)$ is harmonic. Thus the problem is possible.

2°) $\dfrac{\partial u}{\partial x} = 2(1-y) = \dfrac{\partial v}{\partial y}, \Longleftrightarrow v = 2y - y^2 + \underline{\underline{C(x)}}$!

Function

$$\frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x} \quad \Big| \quad \frac{\partial v}{\partial x} = C'(x)$$

$$\Longrightarrow C'(x) = 2x, \quad C(x) = x^2 + \underset{\text{constant}}{C}$$

Thus $v = 2y - y^2 + x^2 + C$

Indeed $f(z) = i z^2 + 2z + iC$

$x = \frac{z+\bar{z}}{2}$

$y = \frac{z-\bar{z}}{2i}$

<u>Analytic functions</u> USUAL RULES:

$$(f+g)' = f' + g', \quad (fg)' = f'g + fg',$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad (g(z) \neq 0)$$

$$\frac{dF(f(z))}{dz} = F'(f(z))\, f'(z) \quad \text{Chain Rule.}$$

<u>Ex:</u> $\frac{d}{dz} e^{(z^2+z+1)} = e^{z^2+z+1}(2z+1)$

__Polynomials__ $\quad P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

$$= a_n (z - z_1)(z - z_2) \cdots (z - z_n)$$

Roots

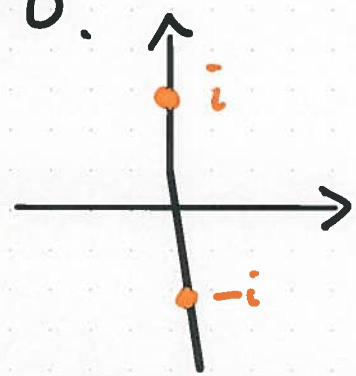
__Rational Functions__

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

is analytic, when $Q(z) \neq 0$.

(m zeros)

__Ex:__ $\dfrac{z}{1+z^2}$ is analytic, when $z \neq \pm i$



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e^z, $\sin(z)$, …, $\sinh(z)$, $\log(z)$, …,

$e^{1/z} \cos(z^3+1)$, … __etc__

TRIGONOM.
HYPERBOLIC.
EXPONENTIAL.
LOGARITHM
etc.

SOME ANALYTIC FUNCTIONS

$$\boxed{e^z = e^x(\cos y + i \sin y),} \qquad z = x+iy$$

$$\frac{de^z}{dz} = e^z, \quad |e^z| = e^x, \quad e^z \neq 0$$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

$$e^z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n \qquad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{z+2\pi i} = e^z \quad (\text{period } 2\pi i)$$

$e^{\pi\sqrt{163}} = 262537412640768743{,}9999999999995\ldots$

$3^3+4^4+3^3+5^5 = ?$

$\frac{355}{113} = ?$

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

$$\boxed{\sin^2 z + \cos^2 z = 1}$$

THE DIFFERENTIATION RULES ARE THE FAMILIAR FORMULAS $(\sin z)' = \cos z$, …

$$\begin{cases} \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1 \\ \cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \end{cases}$$

$$\sin(z+2\pi) = \sin z \quad (\text{period } 2\pi)$$

! $|\sin x| \le 1$ but it is possible that $|\sin z| > 1$.

$$\sin z = z - \frac{z^3}{\ } + \frac{z^5}{\ } - \cdots$$

$$\boxed{\ln z = \ln|z| + i \arg(z)}$$

Examples $\ln(-1) = i(\pi + 2n\pi)$

$\ln(i) = i\left(\frac{\pi}{2} + 2n\pi\right)$

$\ln 1 = 0 + 2n\pi i$ Usually, we take $n = 0$.

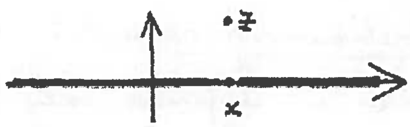
$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \quad (\text{modulo } 2\pi i)$$

DEF.: $a^z = e^{z \ln a} \quad (a \neq 0)$.

Examples $i^i = e^{i \ln i} = e^{-\frac{\pi}{2}} e^{\pm 2n\pi}$

$\sqrt[n]{z} = z^{\frac{1}{n}} = e^{\frac{1}{n}\ln z}$

REMARK The only analytic function which coincides with e^x on the real axis is e^z. No other function will do. More generally, we have the PRINCIPLE OF ANALYTIC CONTINUATION: Suppose that the functions $f(z)$ and $g(z)$ are analytic and that $f(x) = g(x)$ on the real axis. Then $f(z) = g(z)$ (in the domain of analyticity).



¿ $1 = (-1)^2 = \sqrt{(-1)^2} = \sqrt{-1\cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1$?

¿ $2\log(-1) = \log(-1) + \log(-1) = \log[-1\cdot -1] = \log 1 = 0 \Rightarrow$
$\log(-1) = 0 \Rightarrow e^0 = -1$. But $e^0 = +1$, as we know! ?

CAUTION $\boxed{(a^b)^c = a^{bc}}$ May yield the wrong branch.

$$(e^z)^2 = e^z \cdot e^z = e^{2z}$$

$$e^{i\theta} = e^{2\pi i \frac{\theta}{2\pi}} = \left(e^{2\pi i}\right)^{\frac{\theta}{2\pi}} = 1^{\frac{\theta}{2\pi}} = 1$$

$$\boxed{a^b = e^{b \log a}}$$

$\underline{\theta = \pi}$. $-1 = e^{i\pi} = e^{2\pi i \cdot \frac{1}{2}} = \left(e^{2\pi i}\right)^{1/2} = 1^{1/2} = 1$ ↯

$$1^{1/2} = e^{\frac{1}{2}\ln(1)} = e^{1/2[2n i \pi]} = e^{in\pi} = \begin{cases} 1 \\ -1 \end{cases}$$

$\sqrt{1} = \pm 1$

Addendum

$$\frac{d}{dz}\log(z) = \frac{1}{z} \quad (z \neq 0)$$

$$\log(z) = \log r + i\theta$$