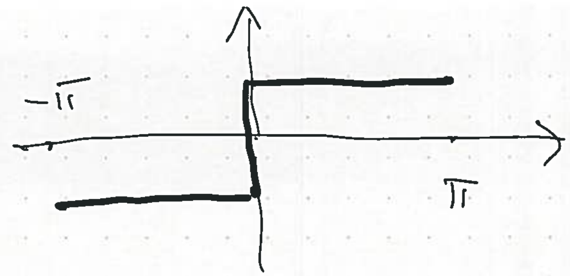
To find the formula for $c_n$, start with

- $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad |x| \le \pi$
- Multiply by $e^{-imx}$
- Integrate termwise.

$$\underline{\underline{\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx}} = \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} \underbrace{e^{inx} e^{-inx}}_{e^{i(n-m)x}}\,dx}_{\begin{cases} 0, & \text{if } n \neq m \\ 2\pi, & \text{if } n = m \end{cases}}$$

$$= \underline{\underline{2\pi c_m}} \qquad \square$$

<u>Ex.:</u> $f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$




$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$\underline{c_0 = 0}$

$$= \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\, dx - \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}\, dx$$

$\underline{n \neq 0}$

$$= \frac{1}{2\pi}\Big/_{0}^{\pi} \frac{e^{-inx}}{(-in)} - \frac{1}{2\pi}\Big/_{-\pi}^{0} \frac{e^{-inx}}{(-in)}$$

$$= \cdots \begin{cases} \frac{-2i}{n\pi} = \frac{2}{n\pi i} & (n = \text{odd}) \\ 0, & n = \text{even} \neq 0. \end{cases}$$

$$f(x) = \sum_{n=\text{ODD}} \frac{2}{\pi i} \frac{e^{inx}}{n} \qquad (|x| < \pi,\ x \neq 0)$$

How to get the answer in sines and cosines?

"Transcription":

$$\frac{2}{\pi i} \sum_{n=\text{ODD}} \frac{e^{inx}}{n} = \frac{2}{\pi i} \Bigg\{ \overset{|n|=1}{\frac{e^{ix} - e^{-inx}}{1}} + \overset{|n|=3}{\frac{e^{3ix} - e^{-3ix}}{3}}$$

$$+ \cdots \Bigg\} = \frac{4}{\pi} \sum_{\substack{n=1 \\ n=\text{ODD}}}^{\infty} \frac{\sin(nx)}{n}. \qquad \blacksquare$$

# PARSEVALS IDENTITY
# (BESSEL'S INEQ.)
# MEAN SQUARE ERROR

**PROBLEM**

TRIGONOMETRIC SERIES

$$\underset{\substack{A_0, A_1, \dots, A_N \\ B_1, B_2, \dots, B_N}}{\text{MINIMUM}} \int_{-\pi}^{\pi} \Big| f(x) - \underbrace{\Big(A_0 + \sum_{n=1}^{N} A_n \cos(nx) + B_n \sin(nx)\Big)}_{\sum_{n=-N}^{N} C_n e^{inx}} \Big|^2 dx$$

FINITE NUMBER OF TERMS

**SOLUTION:** Take $A_0 = a_0$, $A_1 = a_1, \dots, A_N = a_N$, $B_1 = b_1, \dots\dots, B_N = b_n$ or, in other words, $C_n = c_n$. The true Fourier coefficients!

Derivation of, say $A_0 = a_0$.

$$\frac{\partial}{\partial A_0}\int_{-\pi}^{\pi}\left(f(x) - A_0 - \sum_{n=1}^{N} A_n \cos(nx) + B_n \sin(nx)\right)^2 dx \underset{\substack{\uparrow \\ \text{(NECESSARY) CONDITION FOR MINIMUM}}}{=} 0$$

$$\cancel{2}\int_{-\pi}^{\pi}\left(f(x) - A_0 - \sum_{1}^{N}\cdots\right)^1 (-\cancel{1})\, dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\,dx - 2\pi A_0 - \sum_{n=1}^{N}\underbrace{\int_{-\pi}^{\pi}(A_n \cos(nx) + B_n \sin(nx))\,dx}_{=0\ (!)} = 0$$

$$\Longrightarrow A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \underset{\substack{\uparrow \\ \text{Def.}}}{=} a_0 \qquad \blacksquare$$

Please, do $B_3$ yourself. First $\frac{\partial}{\partial B_3} \int = 0$. #

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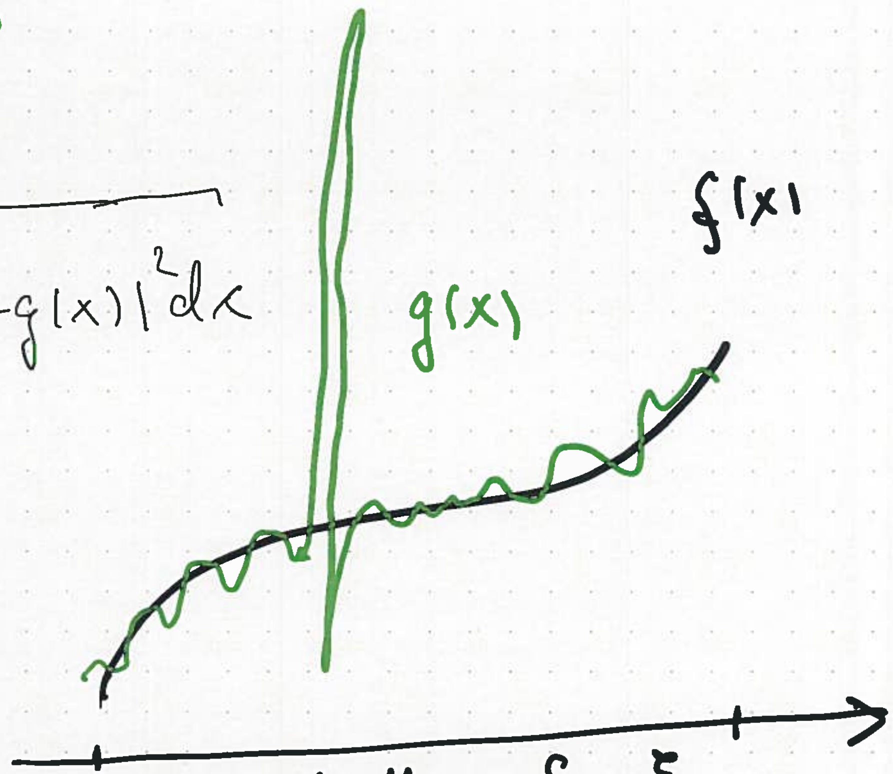
$$\frac{1}{b-a} \int_a^b |f(x) - g(x)|^2 \, dx = \text{MEAN SQUARE ERROR.}$$

approximation $g(x) \sim f(x)$

<u>Remark:</u> The quantity

$$\sqrt{\frac{1}{b-a} \int_a^b |f(x) - g(x)|^2 \, dx}$$

is more natural.




Small Mean Sq. E.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx - \sum_{-N}^{N} |c_n|^2 - \sum_{-N}^{N} |c_n|^2 +$$

$$+ \sum_{-N}^{N} \sum_{-N}^{N} \frac{1}{2\pi} c_n \overline{c_m} \underbrace{\int_{-\pi}^{\pi} e^{i(n-m)x} dx}_{= \begin{cases} 0, & n \neq m \\ 2\pi, & n = m \end{cases}}$$

ORTHOGONALITY

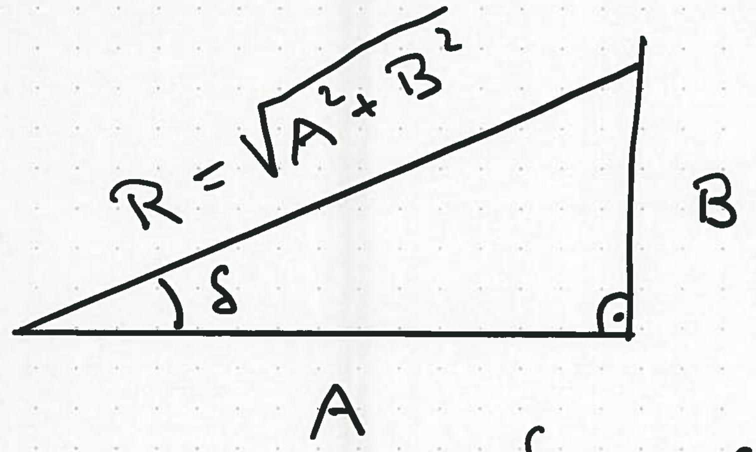
The double sum reduces to $\sum_{-N}^{N} |c_n|^2$.

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 - \sum_{n=-N}^{N} |c_n|^2 \qquad \blacksquare$$

Remark: We also have

$$E_N = \sum_{|n| \geq N+1} |c_n|^2 = \frac{1}{2} \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2).$$

[ **Addendum:** (another chapter; not Parseval)




$$A\cos t + B\sin t$$

$$= \underbrace{R}_{\text{AMPLITUDE OF WAVE}}\cos(t - \underbrace{\delta}_{\text{Phase}})$$

$$\begin{cases} \cos\delta = \dfrac{A}{R} \\ \sin\delta = \dfrac{B}{R} \end{cases}$$

$$A\cos t + B\sin t = R[\underbrace{\cos\delta\cos t + \sin\delta\sin t}_{\cos(t-\delta)}]$$

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