MISCELLANIES

$$z = x + iy = re^{i\theta}$$

$$|z_1 z_2| = |z_1||z_2|$$

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

2 - dimensions .

3 - dimensions NO WAY

4 - dimensions QUATERNIONS

$$\begin{cases} q = x + iy + ju + kv \\ i^2 = j^2 = k^2 = -1 \\ ij = k, \quad jk = i, \quad ki = j \\ ji = -k \quad \text{etc} \end{cases}$$

__Ex.:__ $\left|\dfrac{2z-1}{2-z}\right| \overset{?}{=} 2$ where $|z| = 1$, $z\bar{z} = 1$

2.

$$ {}^2 = \frac{(2z-1)(2\bar{z}-1)}{(2-z)(2-\bar{z})} = \frac{4z\bar{z}-2(z+\bar{z})+1}{4-2(z+\bar{z})+z\bar{z}}$$

$$= \frac{4\cdot 1-2(z+\bar{z})+1}{4-2(z+\bar{z})+1} = 1.$$

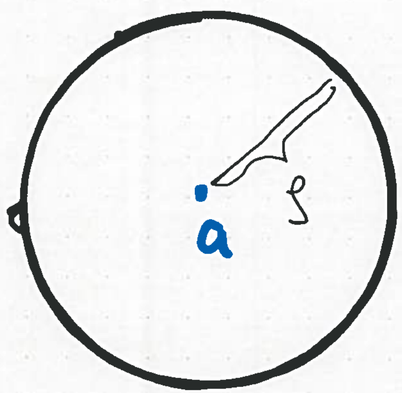
Answer: 1

TWILIGHT ZONE

$$\boxed{i8 = \infty}$$

$\left(i = e^{i\frac{\pi}{2}}\right)$

GEOMETRY



$|z-a| = \rho$

CIRCLE

$|z-a| < \rho$

DISC
(DISK)

$r < |z-a| < R$

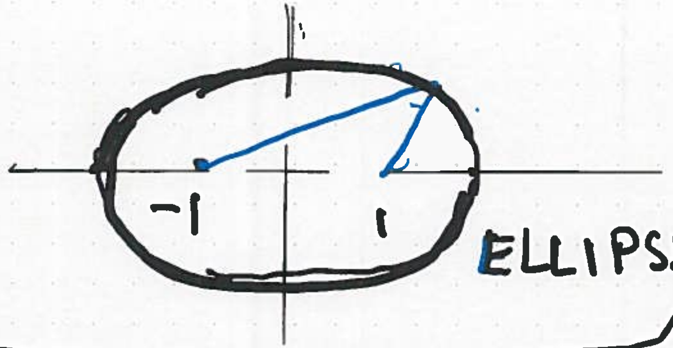
RING
(ANNULUS)

$z \neq a$

$0 < |z-a| < R$

PUNCTURED
DISC

$|z-1| + |z+1| = 3$



ELLIPSE

Ex.: $|z-1||z+1| = 1$

$|z^2-1| = 1, \quad z = re^{i\theta}$

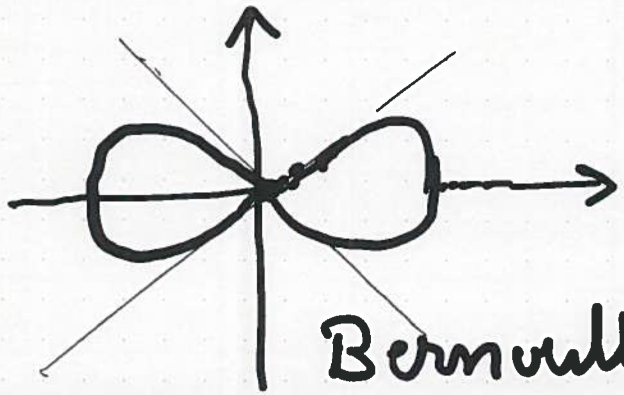
$z \neq 0$

$|r^2 e^{i2\theta} - 1| = 1$

$(r^2 e^{2i\theta} - 1)(r^2 e^{-2i\theta} - 1) = 1$

$r^4 - r^2(e^{2i\theta} + e^{-2i\theta}) + \cancel{1} = \cancel{1}$

$r^2 = 2\cos(2\theta)$



Bernoulli's lemniscate.

ANALYTIC FUNCTIONS

$$f(z) = u(x,y) + i\,v(x,y), \quad z = x+iy$$

<u>Ex:</u> $f(z) = z^2 = (x+iy)^2$

$$= \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$$

LIMITS.

DEF. $\lim\limits_{z\to a} f(z) = A \overset{\text{Def.}}{\iff} \lim\limits_{z\to a} \left| f(z) - A \right| = 0$

This requires that $f(z)$ is defined (at least) when $0 < |z-a| < \rho$ (some small radius).

$z \neq a$ punctured disc.

Furthermore, $\lim_{z \to a} f(z) = \infty \overset{\text{DEF.}}{\Longleftrightarrow} \lim_{z \to a} |f(z)| = \infty$

Ex. $\displaystyle \lim_{z \to i} \frac{z^2+1}{z-i} = \lim_{z \to i} \frac{\cancel{(z-i)}(z+i)}{\cancel{z-i}} = 2i$

NOTE

$$|f(z) - C| = \sqrt{(u(x,y)-A)^2 + (v(x,y)-B)^2}$$

$C = A + iB$

$\xrightarrow[z \to a]{} 0 \Longleftrightarrow$

$u(x,y) \xrightarrow[z \to a]{} A$ & $v(x,y) \xrightarrow[z \to a]{} B$.

CONTINUITY: $\lim_{z \to a} f(z) = f(a)$

$\overset{\text{DEF}}{\Longleftrightarrow}$ $f(z)$ is continuous at the point a.

This requires that $f(z)$ is defined in some neighbourhood $|z-a| < \rho$. (Also $f(a)$ included.)

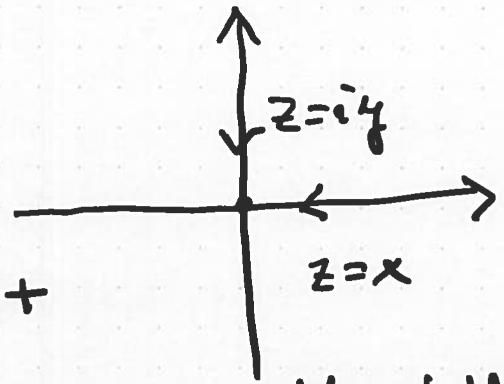
<u>Ex</u>

$$g(z) = \begin{cases} \dfrac{z^2+1}{z-i}, & z \neq i \\ 2i, & z = i \end{cases}$$

is continuous (also) at the point $z = i$.

<u>Ex.</u>

$$f(z) = \frac{z}{|z|}, \quad z \neq 0$$



$$f(x) = \frac{x}{|x|} \to 1, \text{ if } x \to 0+$$

$$f(iy) = \frac{iy}{|iy|} \to i, \text{ if } y \to 0+$$

↯

NO LIMIT

Usual rules for

$$f \pm g, \quad fg, \quad \frac{f}{g}, \quad F(f(z)).$$

DIFFERENTIATION

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad [\text{DEF.}]$$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\Delta z = \Delta x + i \Delta y$$

Notation $\frac{df}{dz}$.

<u>Ex</u>: $f(z) = z^2$ $$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

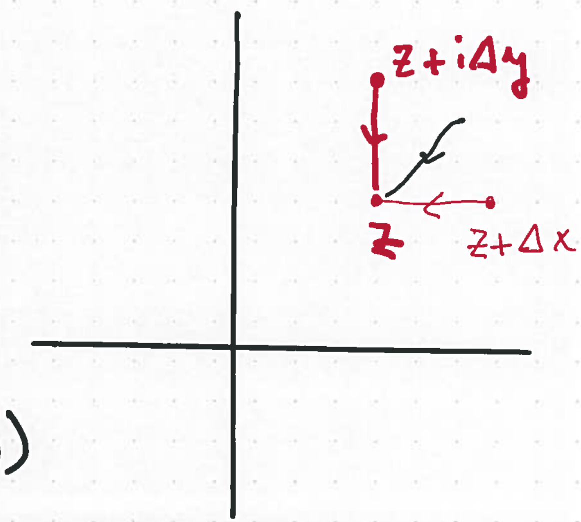
$$= \frac{2z\cdot\Delta z + (\Delta z)^2}{\Delta z} = 2z + \Delta z \xrightarrow[\Delta z \to 0]{\text{as}} 2z$$

$$\therefore \boxed{\frac{dz^2}{dz} = 2z.}$$

<u>Ex.:</u> $g(z) = \bar{z} = x - iy$

$$\frac{g(z+\Delta z) - g(z)}{\Delta z} = \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



1°) <u>$\Delta z = \Delta x$</u> We get

$$\frac{\Delta x - i0}{\Delta x + i0} = \underline{\underline{1}} \quad (\Delta x \to 0)$$

2°) $\underline{\Delta z = i\Delta y}$ We get

The limit depends on the direction in this ex.

$$\frac{0 - i\Delta y}{0 + i\Delta y} = \underline{\underline{-1}} \quad (\Delta y \to 0)$$

$1 \neq -1$

Hence $g(z) = \bar{z}$ is **not** differentiable at any point at all!

- As a rule of thumb, $f(z)$ should be a function of z alone, not of $\bar{z}$ or $\underbrace{|z|}_{\sqrt{\bar{z}z}}$.

ANALYTIC (HOLOMORPHIC) FUNCTIONS

Def.: If $f(z)$ is differentiable at every point in a domain Ω (in the complex plane), then we say that $f(z)$ is an ANALYTIC FUNCTION in Ω.

HOLOMORPHIC

2°) $\underline{\Delta z = i\Delta y}$ We get

$$\frac{0 - i\Delta y}{0 + i\Delta y} = \underline{\underline{-1}} \quad (\Delta y \to 0)$$

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HOLOMORPHIC

1789–1857 1826–1866

CAUCHY–RIEMANN EQNS.

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

[CAUCHY–RIEMANN EQ.]

<u>Ex.:</u> $f(z) = z^2 = x^2 - y^2 + 2xy\,i$

$u = x^2 - y^2$, $\dfrac{\partial u}{\partial x} = 2x$, $\dfrac{\partial u}{\partial y} = -2y$

$v = 2xy$, $\dfrac{\partial v}{\partial x} = 2y$, $\dfrac{\partial v}{\partial y} = 2x$

The Cauchy–Riemann eqs are valid, indeed.

DEDUCTION OF THE C-R EQNS FOR AN ANALYTIC FUNCTION.

Assume: $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

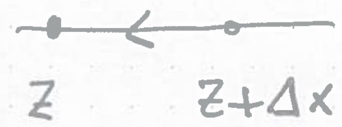
$f = u + iv$

1°) $\Delta z = \Delta x$

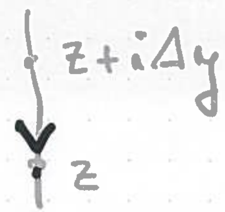
$$f'(z) = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \lim_{\Delta x \to 0} \frac{iv(x+\Delta x, y) - iv(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$



2°) $\Delta z = i\Delta y$



$$f'(z) = \lim_{i\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{i v(x, y+\Delta y) - i v(x,y)}{i\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

3°) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ $\left(-i = 1/i\right)$

Real Part

Imaginary Part.

These ARE the CAUCHY–RIEMANN EQS.