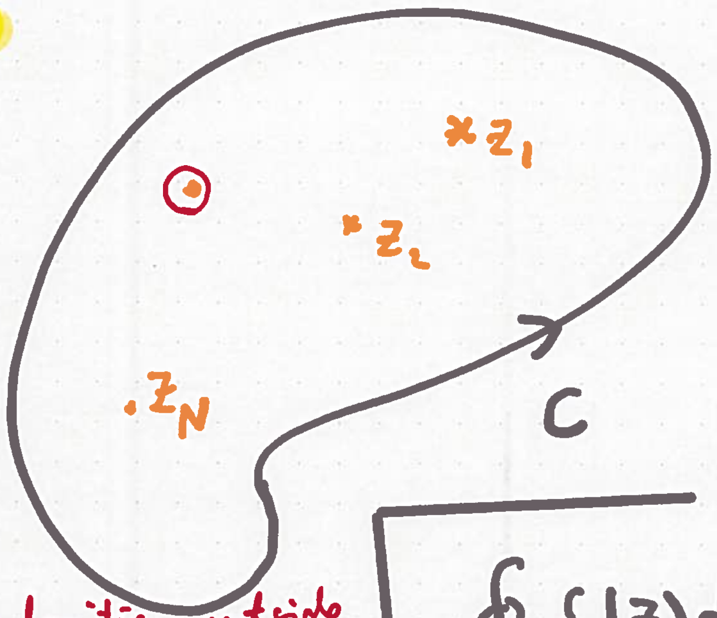
RESIDUE THEOREM



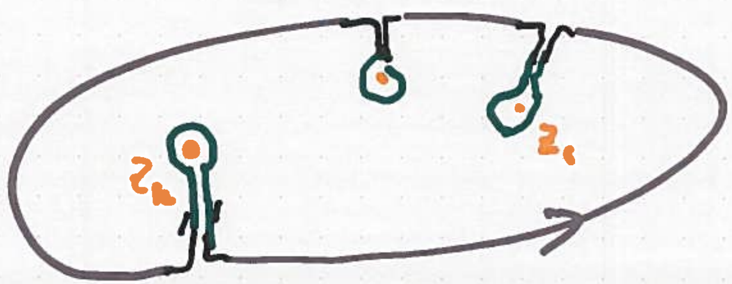
$f(z)$ is analytic inside and on the contour C with the exception of the isolated singularities $z_1, z_2, \ldots, z_N$ (inside).

Singularities outside do not count. None on the contour.

Then —

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z=z_k}\{f(z)\}}$$

Proof:



"Key holes"

$$0 \overset{\text{CAUCHY}}{=} \int_{\text{Key Hole}} f(z)\,dz = \oint_C f(z)\,dz + \sum_{k=1}^{N} \oint_{|z-z_k|<\delta} f(z)\,dz$$

[Small Circles]

Clockwise

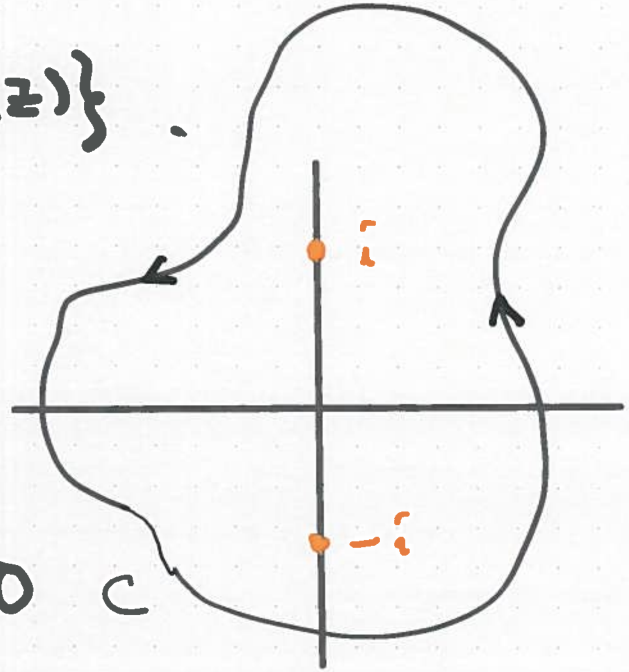
$$= \oint_C f(z)\,dz - \sum_{k=1}^{N} \oint f(z)\,dz$$

Counter clockwise

$$= \oint_C f(z)\,dz - 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z=z_k} \{ f(z) \} .$$

<u>Ex.:</u>

$$\oint_C \frac{dz}{z^2+1} = 2\pi i \left\{ \operatorname{Res}_{z=i} \frac{1}{1+z^2} + \operatorname{Res}_{z=-i} \frac{1}{1+z^2} \right\} = \cdots = 0$$



How to calculate residues.

1) SIMPLE POLE z_0

- $$\boxed{\operatorname*{Res}_{z=z_0}\{f(z)\} = \lim_{z\to z_0}\left[(z-z_0)f(z)\right]}$$

Proof.: $$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots$$

$$\longrightarrow a_{-1} \quad \text{as} \quad z \to z_0. \quad \square$$

- $$f(z) = \frac{p(z)}{q(z)}, \quad q(z_0) = 0, \; q'(z_0) \neq 0, \; p(z_0) \neq 0$$

$$\boxed{\operatorname*{Res}_{z=z_0}\{f(z)\} = \frac{p(z_0)}{q'(z_0)}}$$

Proof: $$(z - z_0)\frac{p(z)}{q(z)} = \frac{p(z)}{\frac{q(z) - q(z_0)}{z - z_0}}$$

$$\longrightarrow \frac{p(z_0)}{q'(z_0)} \text{ as } z \to z_0 \quad \square \qquad \square$$

2) DOUBLE POLES

$$\operatorname*{Res}_{z \to z_0}\{f(z)\} = \lim_{z \to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$$

$$f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \cdots$$

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = a_{-1} + a_0(z-z_0) + \cdots$$

$$\longrightarrow a_{-1} \text{ as } z \longrightarrow z_0$$

3) ESSENTIAL SING.

?? Use the Laurent expansion.

Ex.: $f(z) = \dfrac{1}{1+z^2} = \dfrac{1}{(z-i)(z+i)}$ $z = \pm i$ are simple poles.

$$\operatorname*{Res}_{z=i}\left\{\frac{1}{1+z^2}\right\} = \lim_{z \to i} \frac{\cancel{(z-i)}}{\cancel{(z-i)}(z+i)} = \frac{1}{i+i} = \frac{1}{2i}$$

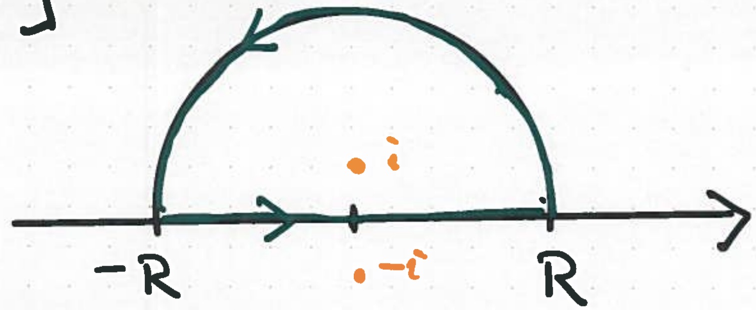
Ex: $\dfrac{\sin(z^2)\,(e^z-1)\,e^{\cos(z)}}{(z^2+1)^2\, e^{-z^2}\, z^4\,(z-2)}$ $\dfrac{h(z)}{g(z)}$

- $z=2$ is a simple pole
- $z=\pm i$ are double poles
- $z=0$ is a SIMPLE POLE

$\dfrac{\sin(z^2)}{z^2}\;\dfrac{e^z-1}{z}\;\dfrac{1}{z}\cdots$

$z \approx 0$

<u>Ex:</u> $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = ?$ [Answer $= \pi$]



simple pole

$$\oint_{D} \frac{dz}{1+z^2} = 2\pi i \operatorname*{Res}_{z=i}\left\{\frac{1}{1+z^2}\right\}$$

$$= 2\pi i \cdot \lim_{z\to i}\left(\frac{z-i}{z^2+1}\right) = \frac{2\pi i}{2i} = \pi$$

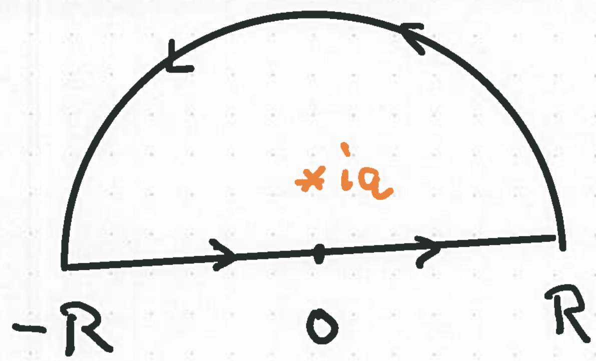
$$\boxed{\text{In fact, we know } \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}}$$

$$\pi = \int_{D} = \int_{-R}^{R} \frac{dx}{1+x^2} + \underbrace{\int_{\substack{|z|=R \\ y>0}} \frac{1}{1+z^2}\,dz}_{\to 0 \text{ as } R\to\infty.}$$

Indeed, $\displaystyle\left|\int_{\substack{|z|=R \\ y>0}} \frac{dz}{1+z^2}\right| \le \pi R \frac{1}{R^2-1} \longrightarrow 0$ as $R \to \infty$.

Ex. $$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = ? \quad (a>0)$$

$(R > a)$



$$\oint_{\triangle} \frac{e^{iz}}{z^2+a^2}\,dz = 2\pi i \operatorname*{Res}_{z=ia}\left\{\frac{e^{iz}}{z^2+a^2}\right\}$$

$$= 2\pi i \lim_{z\to ia} \frac{\cancel{(z-ia)}\, e^{iz}}{\cancel{(z-ia)}(z+ia)} = 2\pi i \frac{e^{-a}}{2ia}$$

$z^2+a^2=0 \Longleftrightarrow$
$z = \pm ia$
SIMPLE POLES

$$= \frac{\pi e^{-a}}{a}$$

Half-circle $$\left|\int_{\substack{|z|=R\\ y>0}} \frac{e^{iz}\,dz}{z^2+a^2}\right| \le \pi R \frac{1}{R^2-a^2} \longrightarrow 0.$$

$iz = ix - y$

$|e^{iz}| \le e^{-y} \le 1, \quad y>0$

Hence
$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}$$

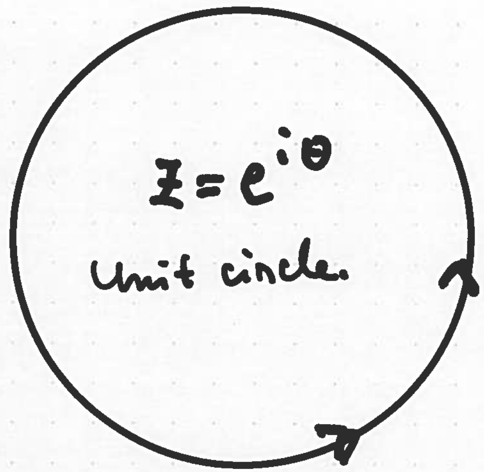
We read off that
$$\int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x^2+a^2} = \frac{\pi e^{-a}}{a}$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)\,dx}{x^2+a^2} = 0$$ (Also evident without our calculations.)

TRIGONOMETRIC INTEGRALS

A rational function of $\cos(\theta)$, $\sin(\theta)$.

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$$
$$= \oint_{|z|=1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{dz}{iz}$$



$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \boxed{z = e^{i\theta}}$$

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta$$

Substitutions:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}, \qquad \sin\theta = \frac{z-\frac{1}{z}}{2i} \qquad d\theta = \frac{dz}{iz}$$

Ex.

$$I = \int_0^{2\pi} \frac{d\theta}{5+3\cos(\theta)} = \oint_{|z|=1} \frac{\frac{dz}{iz}}{5+3\,\frac{z+\frac{1}{z}}{2}}$$

[algebra]

$$= \cdots = -\frac{2i}{3}\oint \frac{dz}{(z+\frac{1}{3})(z+3)} = -\frac{2i}{3}\cdot 2\pi i \cdot \operatorname*{Res}_{z=-\frac{1}{3}}\left\{\frac{1}{(z+\frac{1}{3})(z+3)}\right\}$$

$$= \pi/2$$

Only $-\frac{1}{3}$ is inside the circle $|z|=1$.