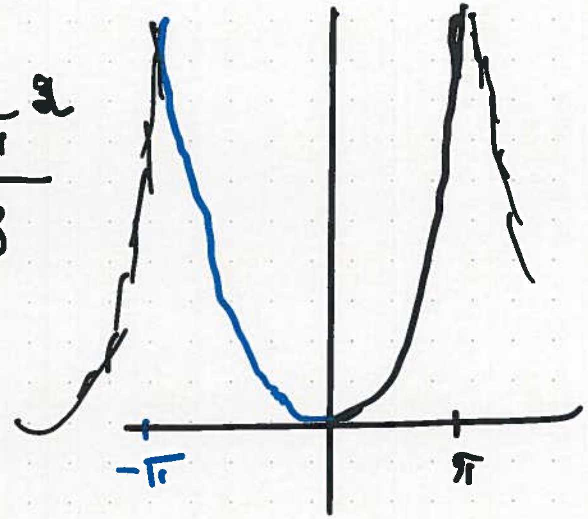
Ex.: The cosine series of $f(x) = x^2$ over the interval $[0, \pi]$ is the Fourier series of the even extension x^2 to $[-\pi, \pi]$. (The corresponding sine series is the Fourier series of the odd extension $x|x|$).

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{3}$$



$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\, dx$$

$$= \cdots = \frac{4(-1)^n}{n^2}, \quad n \geq 1.$$

__Answer:__ $$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n^2}, \quad |x| \leq \pi$$

So what ?

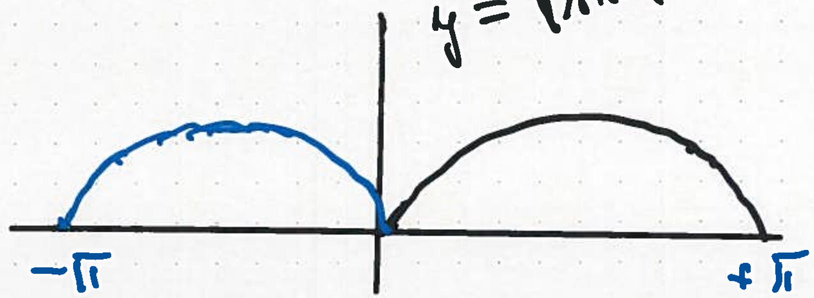
$\underline{x = \pi}$ $\quad \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad [\cos(n\pi) = (-1)^n]$

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$ Euler

$\underline{x = 0}$ Something!

Ex: $f(x) = \sin(x)$, $0 \le x \le \pi$

THE COSINE SERIES



$b_n = 0$

$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin|x|\,dx = \frac{2}{\pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{\sin(x)\cos(nx)}_{\frac{1}{2}\left[\sin((n+1)x) + \sin((1-n)x)\right]}\,dx \qquad a_{odd} = 0$$

$$= \cdots = \frac{4}{\pi}\,\frac{1}{1-n^2} \qquad (n = 2, 4, 6, \dots)$$

$n \longleftrightarrow 2m$

<u>Answer</u>

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{4n^2-1}, \qquad |x| \leq \pi$$

<u>Addendum</u>:

$$\frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{i(n)x} + e^{-inx}}{2} \qquad \text{EULER}$$

$$= \frac{1}{4i}\left[\underbrace{e^{i(n+1)x} - e^{-i(n+1)x}}_{2i\sin(n+1)x} + \underbrace{e^{i(1-n)x} - e^{i(n-1)x}}_{2i\sin((1-n)x)}\right]$$

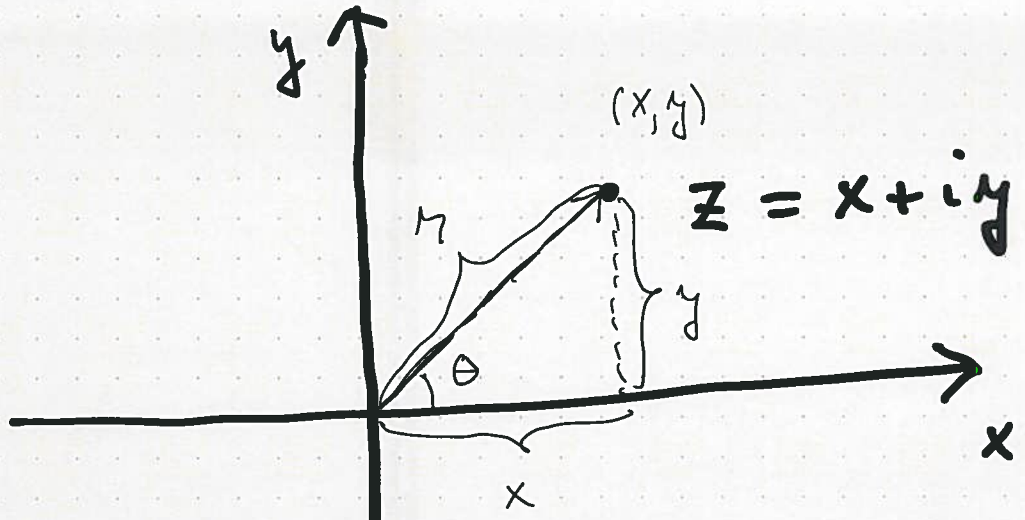
- AN EXAMPLE WITH INTERFERENCE: see Forced spring on the homepage.

COMPLEX FORM

$$a_0 + \sum_{n=1}^{N} a_n \underbrace{\cos(nx)}_{\frac{e^{inx}+e^{-inx}}{2}} + b_n \underbrace{\sin(nx)}_{\text{\%.}} = \sum_{n=-N}^{N} c_n e^{inx}$$

$$\boxed{\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i\sin(\theta) \end{aligned}}$$ EULER

$$\boxed{\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}} \qquad \boxed{\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$$

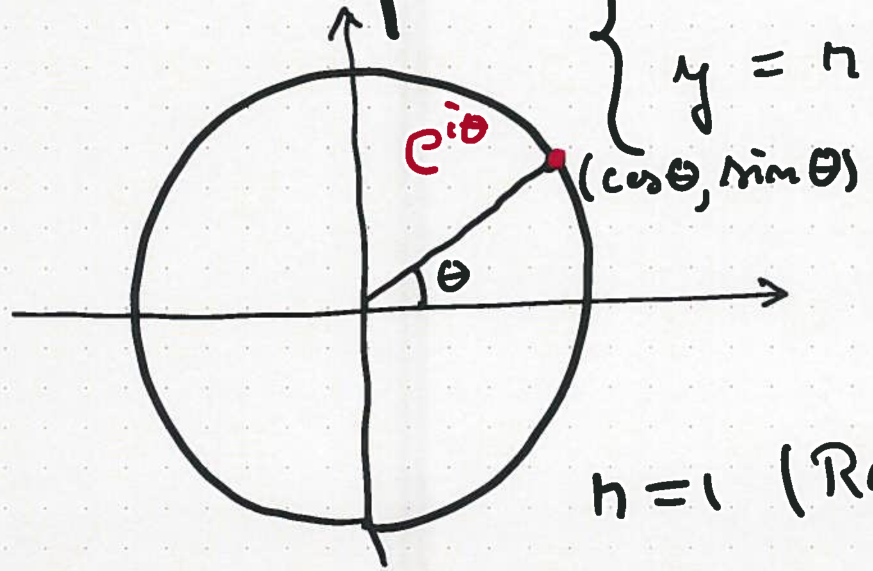


$$z = x + iy$$

$$\boxed{i^2 = -1}$$

$$r^2 = x^2 + y^2$$

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$



$r = 1$ (Radius)

NOTATION

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

Ex

$$a + \sum_{n=1}^{3} \frac{\cos(nx)}{n^2} = \sum_{n=-3}^{3} c_n e^{inx}$$

$c_n = ?$

$$a + \sum_{n=1}^{3} \frac{e^{inx} + e^{-inx}}{2n^2} = a + \frac{e^{ix} + e^{-ix}}{2}$$

$$+ \frac{e^{2ix} + e^{-2ix}}{8} + \frac{e^{3ix} + e^{-3ix}}{18}$$

$c_0 = a$, $c_1 = 1/2$, $c_{-1} = \frac{1}{2}$, $c_2 = 1/8$, $c_{-2} = \frac{1}{8}$

$c_3 = \frac{1}{18}$, $c_{-3} = \frac{1}{18}$. [$c_n = 0$ when $|n| \geq 4$]

#

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \quad (n = 0, \pm 1, \pm 2, \dots)}$$

ORTHOGONALITY

$$e^{-imx}$$

$$\int_{-\pi}^{\pi} e^{inx}\,\overline{e^{imx}}\,dx = \begin{cases} 0, & n \neq m \\ 2\pi, & n = m \end{cases}$$

(n,m integers!)

$$\int_{\alpha}^{\beta} e^{i\lambda x}\,dx = \left.\frac{e^{i\lambda x}}{i\lambda}\right|_{\alpha}^{\beta} \qquad (\lambda \neq 0)$$

$$\boxed{\frac{de^{z}}{dz} = e^{z}}$$