Weierstrass's M-test for uniform convergence.

If

Constants

(i) $|f_n(z)| \le M_n$ when $z \in D$, $n = 1, 2, 3, \dots$

(ii) $\sum_{n=1}^{\infty} M_n < \infty$

then the series $\sum_{n=1}^{\infty} f_n(z)$ converges <u>uniformly</u> in D

Ex.: $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$, $-\infty < x < \infty$.

Claim: The series converges uniformly in $(-\infty, \infty)$.

Proof: 1°) $\left|\frac{\sin(nx)}{n^2}\right| \le \frac{1}{n^2} \ (= M_n)$ when x is real.

2°) $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. By Weierstraß's M-test, the convergence is uniform in $(-\infty, \infty)$

(**Remark:** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nz)}{n^2}$ is divergent, if $\operatorname{Im}(z) \neq 0$.)

Ex A dangerous rearrangement of the conditionally convergent series

$$\ln(2) = s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

$$\begin{cases} s = \overset{\mathrm{I}}{\left(1 - \frac{1}{2}\right)} + \overset{\mathrm{II}}{\left(\frac{1}{3} - \frac{1}{4}\right)} + \overset{\mathrm{IV}}{\left(\frac{1}{5} - \frac{1}{6}\right)} + \overset{\mathrm{IV}}{\left(\frac{1}{7} - \frac{1}{8}\right)} + \overset{\mathrm{V}}{\left(\frac{1}{9} - \frac{1}{10}\right)} + \overset{\mathrm{VI}}{\left(\frac{1}{11} - \frac{1}{12}\right)} + \cdots \\[2ex] \frac{1}{2} s = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \end{cases}$$

$$\frac{3}{2} s = 1 + \cancel{(}\frac{1}{3} - \frac{1}{2}\cancel{)} + \frac{1}{5} + \cancel{(}\frac{1}{7} - \frac{1}{4}\cancel{)} + \frac{1}{9} + \cancel{(}\frac{1}{11} - \frac{1}{6}\cancel{)} + \cdots$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad R = 1$$

1

LAURENT EXPANSION

<u>Ex.:</u>

$$\frac{\sin(z)}{z^2} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots}{z^2}$$

$$= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \dots, \quad z \neq 0$$

LAURENT SERIES

Cannot be written as a Taylor series

$$\frac{\sin(z)}{z^2} = \sum_{n=0}^{\infty} a_n z^n \quad \lightning$$

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

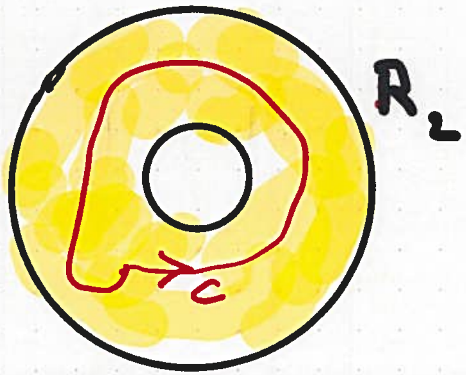
THM Suppose that $f(z)$ is analytic in the ring $R_1 < |z-z_0| < R_2$ (where $0 \le R_1 < R_2 \le \infty$). Then the expansion

RESIDUE

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

is valid, when $R_1 < |z-z_0| < R_2$. Moreover

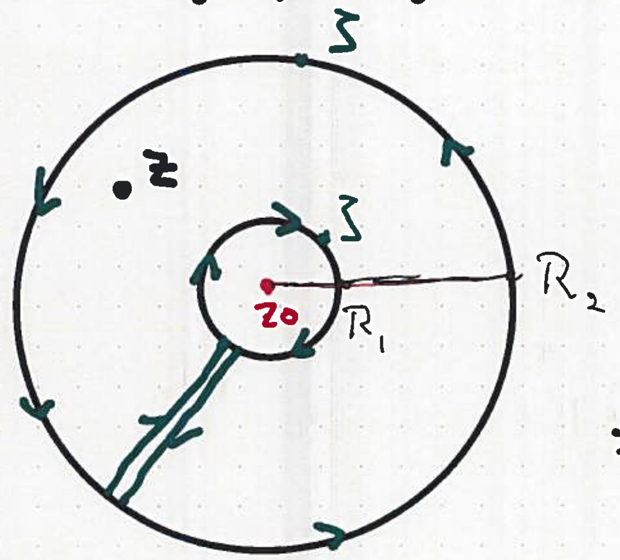
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$



$$\boxed{a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz}$$

RESIDUE AT THE POINT z_0

Idea of proof. Key hole contour



$$f(z) = \frac{1}{2\pi i}\oint_{\Omega} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$= \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R_2^-} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\oint_{|\zeta - z_0| = R_1^+} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^n$$

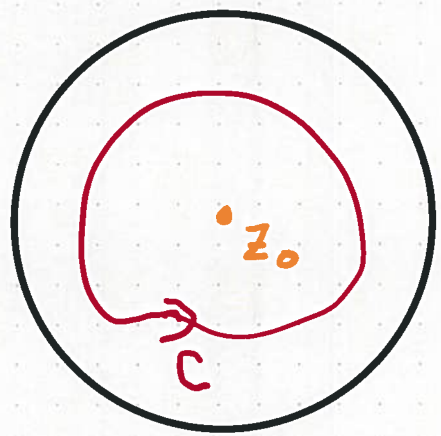
inner circle

<u>Important case:</u> $R_1 = 0$, $f(z)$ is analytic when $0 < |z - z_0| < R$ [A punctured disk].

$z \neq z_0$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

This can be integrated termwise



$$\oint_C f(z)\,dz = \oint_C \Big(\sum_{-\infty}^{\infty} \cdots\Big) dz$$

$$= \sum_{n=-\infty}^{\infty} a_n \oint_C (z - z_0)^n dz = \cdots + 0 + 0 + a_{-1} \cdot 2\pi i + 0 + 0 + \cdots = 2\pi i \, a_{-1}$$

Only one integral survived! "Residue".

ISOLATED SINGULARITIES.

Assume that $f(z)$ is analytic when $0<|z-z_0|<R$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \qquad 0<|z-z_0|<R$$

ISOLATED SINGULARITY.

Three possibilities.

I. $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

REMOVABLE

Ex.: $\dfrac{\sin(z)}{z} = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots, \quad 0<|z|<\infty$

II $f(z) = \dfrac{a_{-n}}{(z-z_0)^n} + \cdots + \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_1 + \cdots$

where $a_{-n} \neq 0$

A POLE of ORDER n

III Infinitely many $a_{-n} \neq 0$, $n \leq -1$.

ESSENTIAL SINGULARITY.

Ex $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots, \quad z \neq 0.$

Ex: $f(z) = \dfrac{1}{z^2 - 1} = \sum_{n=1}^{\infty} a_n (z-1)^n, \quad a_n = ?$

$$\frac{1}{(z-1)(z+1)} = \frac{1}{(z-1)[(z-1)+2]} = \frac{1}{(z-1)2\left[1 + \frac{z-1}{2}\right]}$$

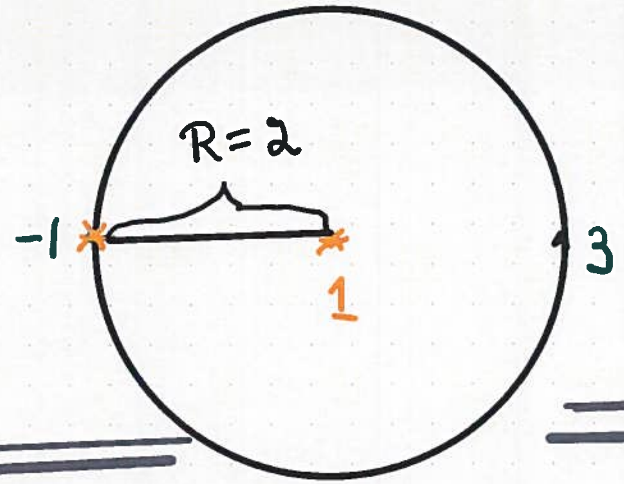
$$= \frac{1}{2(z-1)}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \cdots\right]$$

$$= \frac{1/2}{z-1} - \frac{1}{4} + \frac{1}{8}(z-1) - \cdots, \quad \left|\frac{z-1}{2}\right| < 1$$

A SIMPLE POLE

The RESIDUE at the point $z = 1$ is

$$a_{-1} = \frac{1}{2}$$



ZEROS OF ANALYTIC FUNCTIONS

THE TAYLOR EXPANSION ASSUMED:

$$f(z) = \cancel{a_0} + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

IF $f(z_0) = 0,\ f'(z_0), \ldots, f^{(n-1)}(z_0) = 0$ but

$$f^{(n)}(z_0) \neq 0,$$

we say that z_0 is a zero of <u>order</u> n.

$$f(z) = (z-z_0)^n\Big[a_n + \underbrace{a_{n+1}(z-z_0) + \cdots}_{\to 0 \text{ as } z \to z_0}\Big]$$

where $a_n \neq 0$.

$\Rightarrow [\ \ ] \neq 0$ when $|z-z_0| < \delta$ (some small number)

Hence $\underline{f(z) \neq 0}$ when $\underline{0 < |z-z_0| < \delta}$

Thus the zero z_0 is ISOLATED.

!! The zeros of analytic functions are <u>isolated</u>. !!

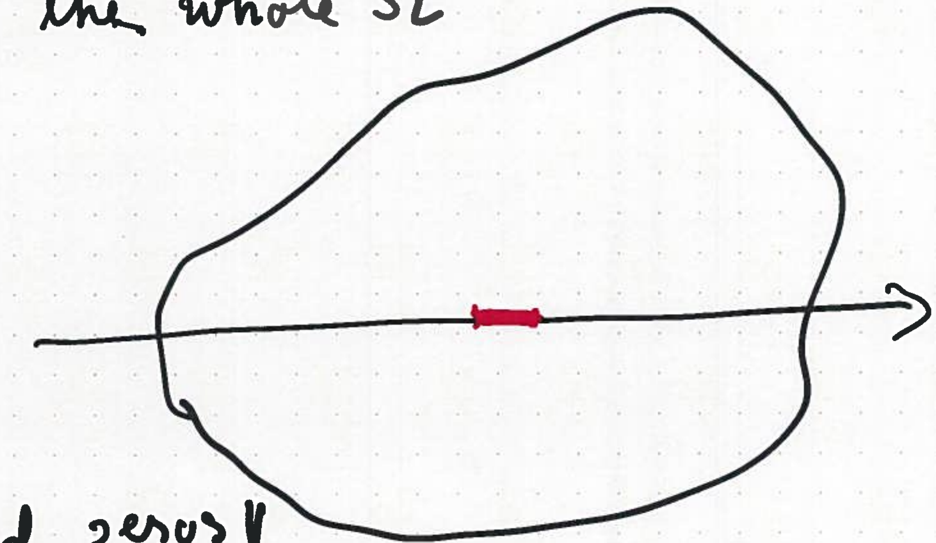
CONSEQUENCE Principle of unique continuation.

Suppose that

1) $f(z)$ and $g(z)$ are analytic in Ω

2) $f(z) = g(z)$ on a segment (curve) in Ω

Then $f(z) \equiv g(z)$ in the whole Ω



Ω

Proof:

$F(z) = f(z) - g(z)$ is analytic

$F(z)$ has non-isolated zeros!

$\Rightarrow F(z) \equiv 0$ $\square$