EXPONENTIAL FUNCTION

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \qquad (R = \infty)$$

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

EULER.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

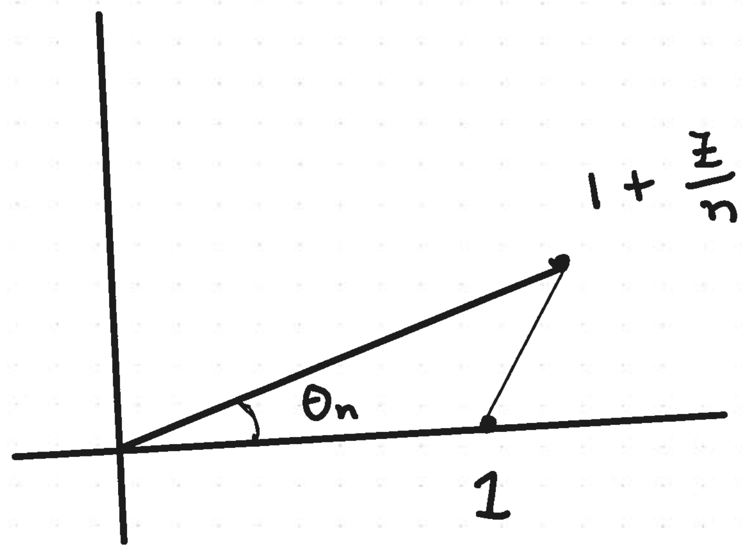
$$= \underbrace{1 - \frac{1}{2!}\theta^2 + \frac{\theta^4}{4!} - \cdots}_{\cos(\theta)} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \cdots\right)}_{\sin(\theta)} \quad \square$$

This proves Euler's formula.

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$\boxed{\lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n = e^z}$$

Proof: We know that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$



$$z = x+iy$$

$$1+\frac{z}{n} = 1+\frac{x}{n}+i\frac{y}{n}$$

$$\theta_n = \overline{\arctan}\left(\frac{y/n}{1+\frac{x}{n}}\right)$$

$$= \frac{y/n}{1+\frac{x}{n}} + O\left(\frac{1}{n^3}\right)$$

$$\left|1+\frac{z}{n}\right|^2 = \left(1+\frac{x}{n}\right)^2 + \frac{y^2}{n^2}$$

$$= 1+\frac{2x}{n}+\frac{x^2+y^2}{n^2}$$

$$1+\frac{z}{n} = \left|1+\frac{z}{n}\right|e^{i\theta_n}$$

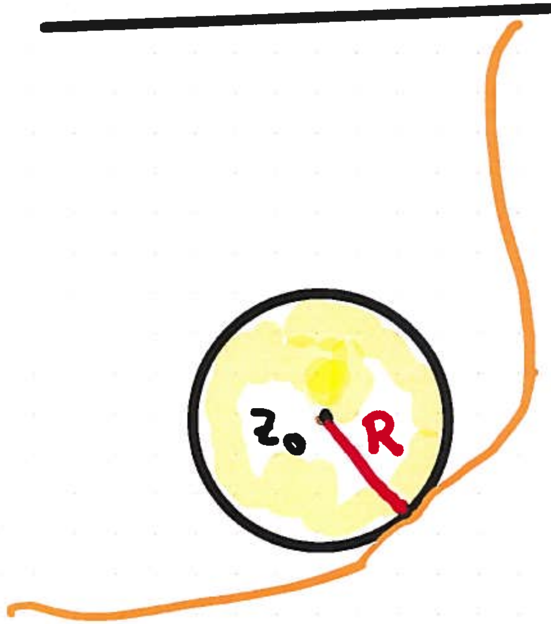
$$\left(1+\frac{z}{n}\right)^n = \left(\left|1+\frac{z}{n}\right|e^{i\theta_n}\right)^n = \left|1+\frac{z}{n}\right|^n e^{in\theta_n}$$

$$= \underbrace{\left(1 + \frac{x}{\left(\frac{n}{2}\right)} + \frac{x^2+y^2}{n^2}\right)^{\frac{n}{2}}}_{\to e^x} \underbrace{e^{in\left[\frac{y/n}{1+\frac{x}{n}} + O\left(\frac{1}{n^3}\right)\right]}}_{\underbrace{e^{i\left[\frac{y}{1+\frac{x}{n}} + O\left(\frac{1}{n^2}\right)\right]}}_{\to e^{iy}}}$$

Has no influence

#

$$= e^x e^{iy} \quad \square$$



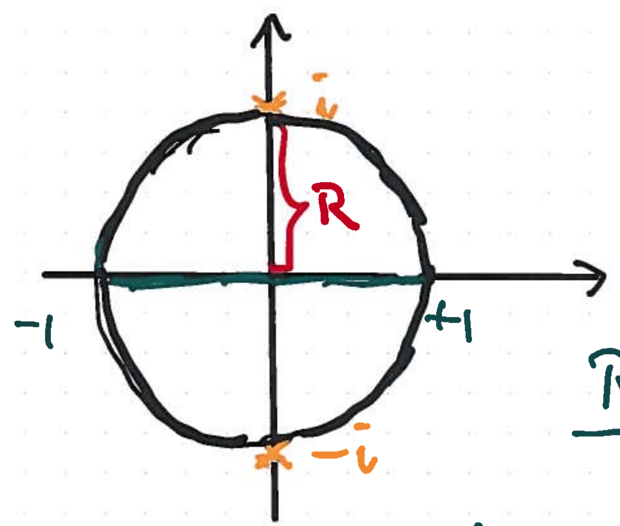
$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{1}{2!} f''(z_0)(z-z_0)^2 + \cdots$$

$$|z - z_0| < R$$
$$= \text{dist}(z_0, \text{boundary of } \Omega)$$

CONVERGENCE IN THE LARGEST DISC IN THE DOMAIN where $f(z)$ is analytic!

__Ex:__ $\dfrac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$

$R = 1$ $\quad 1 + z^2 = 0 \iff z = \pm i$



$f(z) = \dfrac{1}{1+z^2}$ is analytic

when $z \neq \pm i$.

__Remark.__ $\dfrac{1}{1+x^2}$. Then $R = 1$

i.e. $-1 < x < 1$; can be determined by looking at $\dfrac{1}{1+z^2}$.

Ex $\dfrac{1}{\cosh(x)} = 1 + \sum_{n=1}^{\infty} a_n x^n$. Determine R!

so that the series converges when $-R < x < R$, and diverges when $|x| > R$. (Real variables.)

SOLUTION. Go to the complex plane!

$$f(z) = \frac{1}{\cosh(z)} = 1 + \sum_{n=1}^{\infty} a_n z^n$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = 0 \iff e^z = -e^{-z}$$

$$\iff e^{2z} = -1 = e^{i\pi} \iff 2z = i\pi + 2ni\pi$$

$$\iff z = i\frac{\pi}{2} + ni\pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

[$f(z)$ is analytic except at these points]

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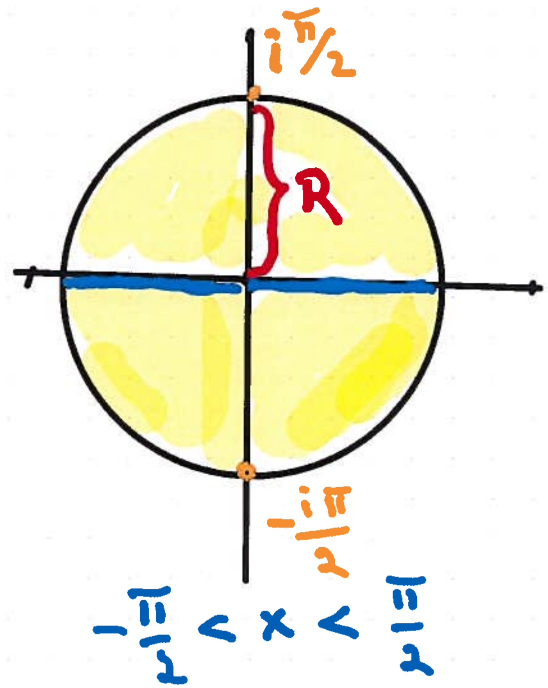


$R = \frac{\pi}{2}$

The series conv. when $|z| < \pi/2$, diverges, when $|z| > \frac{\pi}{2}$.

Answer $R = \frac{\pi}{2}$.

Conclusion: $|a_n| \sim \left(\frac{\pi}{2}\right)^n$ as $n \to \infty$

About the definition of analytic functions.

I $\quad f'(z) = \lim\limits_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

II $\quad f = u + iv$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

CAUCHY-RIEMANN

III $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$ when

Boundary

$$|z-z_0| < R_0 = \mathrm{dist}(z_0, \partial\Omega)$$

In fact $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

A Taylor expansion that converges to the "wrong value".

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad -\infty < x < \infty$$

$$f(0) = 0, \; f'(0) = 0, \; f''(0), \dots, f^{(n)}(0), \dots \; !$$

$$f(x) \overset{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + 0x + 0x^2 + \cdots$$
$$= 0$$

No !

<u>Explanation</u> $e^{-\frac{1}{z^2}}$ is terrible when $z \approx 0$.

UNIFORM CONVERGENCE

<u>Real Variables</u> Suppose that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

at every point x in the interval I. We say that the convergence is <u>uniform</u> <u>in</u> $\underline{I}$ if

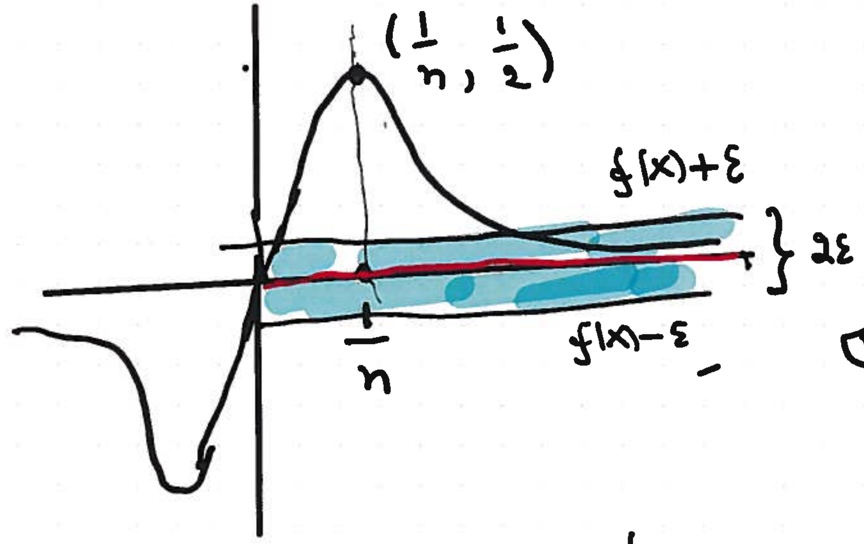
$$\sigma_n = \overset{\text{sup}}{\max_{x\in I}} \left| f(x) - f_n(x) \right| \xrightarrow[n\to\infty]{} 0 .$$

<u>Ex.</u>:

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

The pointwise limit is 0.

$$f(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$$



$$f_n\left(\frac{1}{n}\right) = \frac{1}{2}$$

COMMENT. The point $\frac{1}{n}$ is moving! Escaping.

$$\sigma_n = \max_{x \geq 0} |f_n(x) - f(x)|$$

$$= \left|f_n\left(\tfrac{1}{n}\right) - 0\right| = \frac{1}{2} \not\to 0$$

The conv. is not uniform in the interval $x \geq 0$.

What about $I = (10^{-2020}, \infty)$? Then

$$0 \leq \tilde{\sigma}_n = \max_{x \geq 10^{-2020}} |f_n(x) - 0| \underset{n \geq 10^{2020}}{=} f_n(10^{-2020})$$

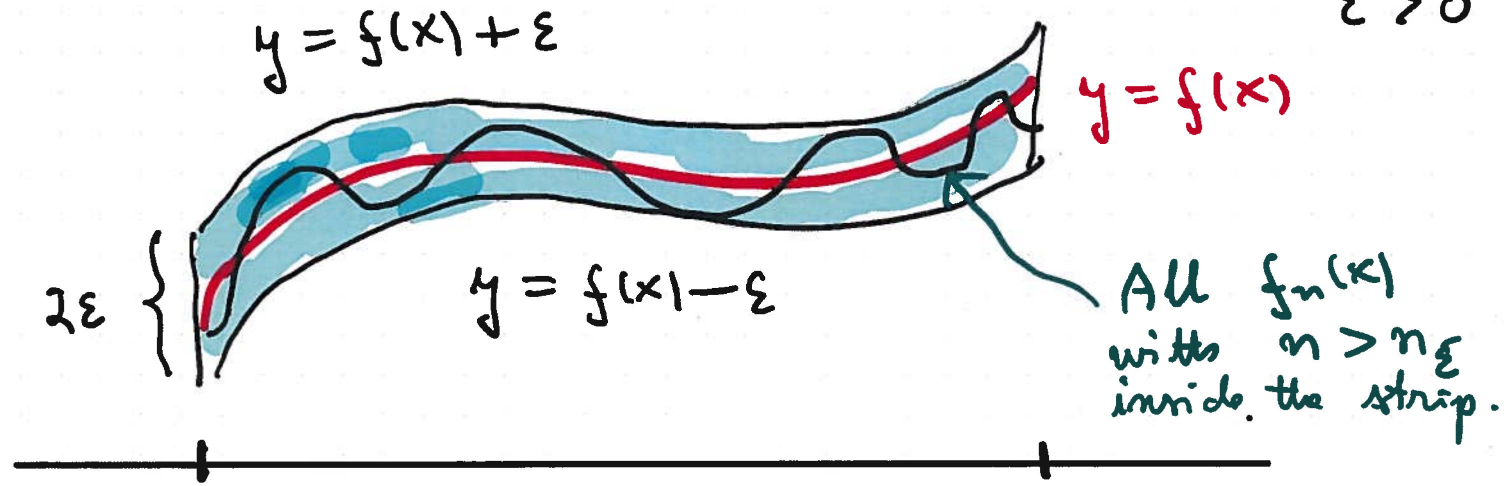
$$= \frac{n \cdot 10^{-2020}}{1 + n^2 (10^{-2020})^2} < \frac{n \cdot 10^{-2020}}{(n \cdot 10^{-2020})^2} = \frac{10^{2020}}{n}$$

$$\longrightarrow 0$$

The convergence is uniform in $[10^{-2020}, \infty)$. #

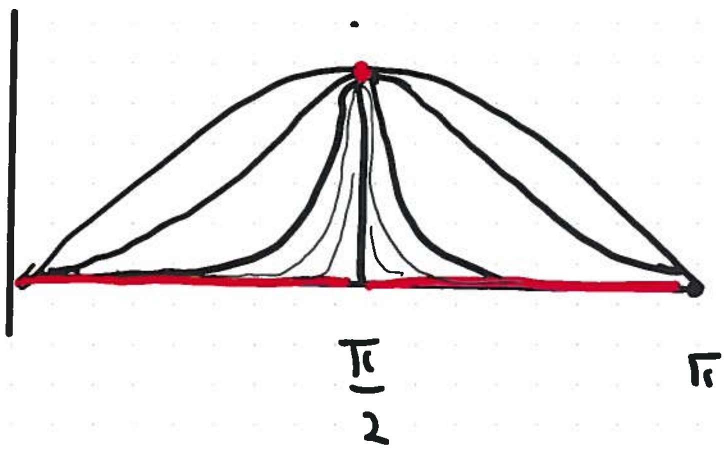
THM Uniform convergence preserves continuity.

$\varepsilon > 0$



<u>Ex:</u> $f_n(x) = (\sin(x))^n \, , \quad 0 \leq x \leq \pi$

$$f(x) = \lim_{n \to \infty} (\sin(x))^n = \begin{cases} 0, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$$



Discontinuous limit of continuous functions.

NOT <u>UNIFORM CONV.</u> IN $[0, \pi]$

COMPLEX CASE

$$\delta_n = \max_{z \in D} |f_n(z) - f(z)| \longrightarrow 0 \overset{\text{DEF}}{\Longleftrightarrow}$$

The convergence is <u>uniform</u> in the region D.

The series $\sum f_n(z)$ is <u>uniformly</u> <u>convergent</u> in the region D, if the sequence

$$S_n(z) = f_1(z) + \cdots + f_n(z)$$

is uniformly convergent in D.

<u>Ex.:</u> $\sum z^n = 1 + z + z^2 + \cdots$ is unif. convergent in the disc $|z| \leq 0,999999$.

$$S_n(z) = 1 + z + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}$$

$$S(z) = \frac{1}{1-z}$$

$$\max_{|z| \le 0{,}999999} |S(z) - S_n(z)| = \max_z \left| \frac{z^n}{1-z} \right|$$

$$\le \frac{0{,}999999^n}{1 - 0{,}999999} \longrightarrow 0 .$$

But the convergence is not uniform in the whole disc $|z| < 1$. #

THM. If a power series converges when $|z| < R$, then it converges uniformly in every fixed strictly smaller disc, say $|z| < R - \delta$.

(If $R = \infty$, take a disc as large as it pleases you.)