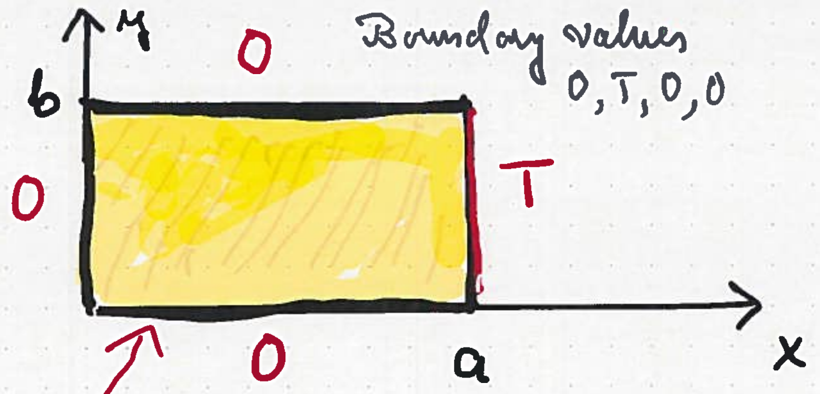
Ex



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

[Kreyszig, pp. 565–566]

$\boxed{u(x,0) = 0, \quad 0 \le x \le a}$

$u(a,y) = T$, $0 \le y \le b$

Constant.

COMMENT. In a bounded domain Ω "the Dirichlet boundary value problem" is

$$\begin{cases} \nabla^2 u = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega \end{cases}$$

Given boundary values

Boundary of Ω

SEPARATION OF VARIABLES.

$$u(x,y) = X(x)\,Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ (Constant)}$$

- $\underline{Y''(y) + \lambda Y(y) = 0}$, $\underline{Y(0) = Y(b) = 0}$

$\underline{\lambda > 0}$ $\quad Y(y) = A \sin(\sqrt{\lambda}\, y) + B \sin(\sqrt{\lambda}\, y)$

$$Y(y) = \text{Const.} \sin\left(\frac{n\pi y}{b}\right), \quad \sqrt{\lambda} = \frac{n\pi}{b}$$

$\left.\begin{array}{l} \lambda = 0 \\ \lambda < 0 \end{array}\right\}$ Nothing useful [Please, verify!]

- $\underline{X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) = 0}$

$$X(x) = a e^{\sqrt{\lambda} x} + b e^{-\sqrt{\lambda} x} \qquad \underline{X_n(x) = A_n' \sinh\left(\frac{n\pi x}{b}\right)}$$

$$\underline{X(0) = 0} \iff a + b = 0$$

It is impossible to satisfy $u(a,y) = T$ for $\underline{\text{one}}$ of these solutions. Use superposition:

SUPERPOSITION

$$u(x,y) = \sum_n A_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

[The values $n=-1,-2,-3,\dots$ are absorbed].

Boundary values

$$T = u(a,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Coefficients in a Fourier sine series!

$$A_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b T \sin\left(\frac{n\pi y}{b}\right) dy = \dots = \frac{2T}{n\pi}\left[1-(-1)^n\right]$$

$$A_n = \begin{cases} 0, & n \text{ even} \\ \dfrac{4T}{n\pi \sinh\left(\frac{n\pi a}{b}\right)}, & n \text{ odd} \end{cases}$$

Answer:

$$u(x,y) = \frac{4T}{\pi} \sum_{\substack{n=1,3,5,7,\ldots \\ \text{ODD}}} \frac{\sinh \frac{n\pi x}{b} \cdot \sin \frac{n\pi y}{b}}{n \sinh\left(\frac{n\pi a}{b}\right)} .$$

Urokkelig som tiden
er tallenes viden.
Deres fletninger er,
i evig morgenskjær,
renere enn sneen,
finere enn luften;
men sterkere enn verden,
som de veier uten skåler
og belyser uten stråler.

--Bjørnstjerne Bjørnson

COMPLEX ANALYSIS

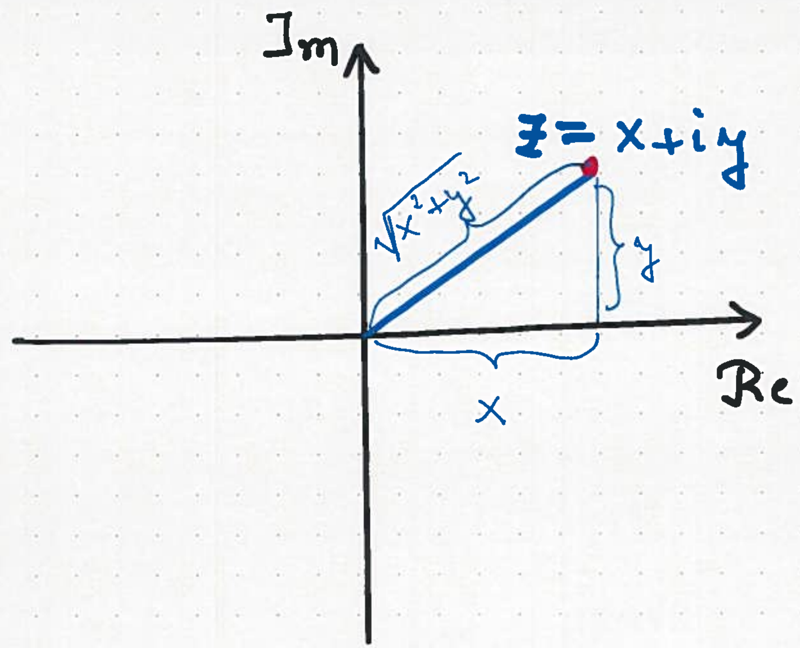
$$\begin{cases} z = x + iy, & i^2 = -1 \\ \bar{z} = x - iy \end{cases}$$

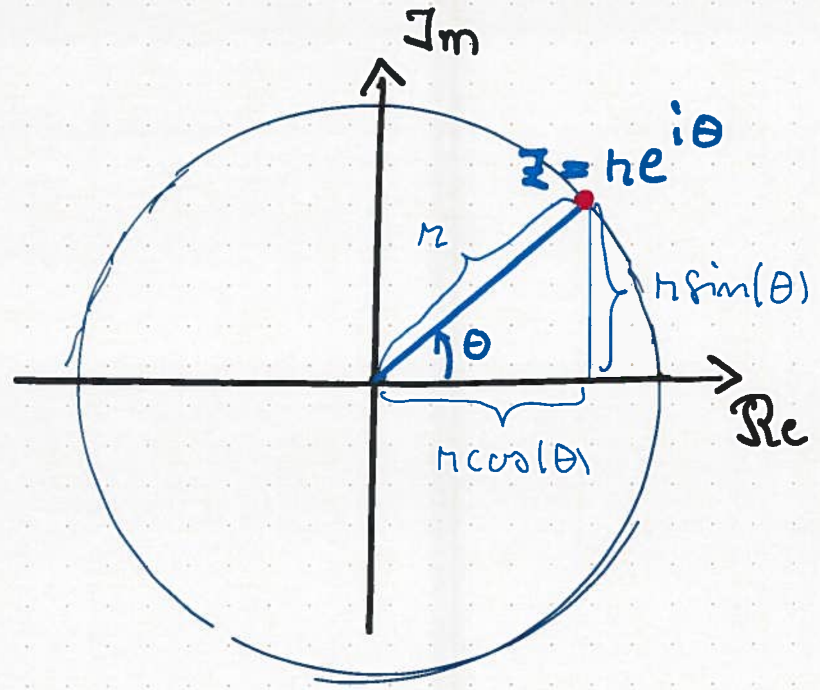
$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\begin{cases} z = re^{i\theta} \\ \bar{z} = re^{-i\theta} \end{cases}$$

$$|z| = r$$





EXTRACTION OF ROOTS

DEF. $z^n = w \iff z = \sqrt[n]{w}$

Ex $\sqrt{i} = \sqrt[2]{i} = \pm \frac{1+i}{\sqrt{2}}$, $\sqrt{1} = \pm 1$

$\sqrt[4]{1} = 1, -1, i, -i.$

Denote $w = \rho e^{i\varphi}$ $(\rho > 0)$ ρ, φ are known.

Assume $w \neq 0$.

$$z = re^{i\theta} \quad (r > 0) \quad r = ?,\ \theta = ?$$

$$z^n = w \iff (re^{i\theta})^n = \rho e^{i\psi}$$

$$\iff r^n e^{in\theta} = \rho e^{i\psi}$$

[(*) One complex eqn is two real eqns.]

$$\overset{(*)}{\iff} \begin{cases} r^n = \rho \ \& \\ n\theta = \psi + 2k\pi \end{cases} \quad (k = 0, \pm 1, \pm 2, \dots)$$

$$\iff \begin{cases} r = \sqrt[n]{\rho} \ \& \\ \theta = \frac{\psi}{n} + \frac{2k\pi}{n} \end{cases}$$

The real pos. root.

$$\boxed{\sqrt[n]{w} = \sqrt[n]{\rho}\, e^{i\left(\frac{\psi}{n} + \frac{2k\pi}{n}\right)}} \quad (k = 0, \pm 1, \pm 2, \dots)$$

Exactly n <u>different</u> values of $\sqrt[n]{w}$!