Addendum*.

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & -\infty < x < \infty \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

"damping", caused by external forces.

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$

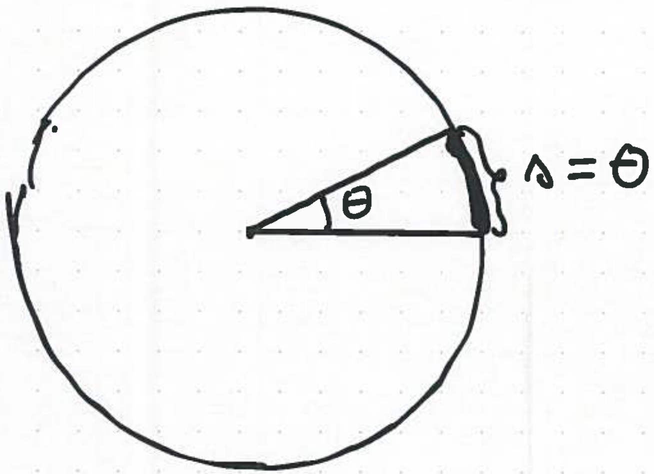
$$+ \frac{1}{2c}\int_0^t \left\{ \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\xi,\tau)\,d\xi \right\} d\tau .$$

[Not in our syllabus 2020]

\#

Ex.: $F(x,t) \equiv 1$. (Easy.!)

Ex FOURIER'S RING (Separation of variables for the Heat Eqn).



Totally insulated ring, radius = 1.

$$u_t = k u_{ss},$$

For convenience.

Write $\boxed{u_t = u_{xx}}$ $(s = x)$

Initial temp.

$$\boxed{u(x,0) = f(x)}$$

Notice $f(x+2\pi) = f(x)$ Periodicity.

Also $u(x,t) = u(x+2\pi, t)$

The ring is closed!

Separation of variables $u(x,t) = X(x)\,T(t)$

$$X\dot{T} = X''T$$

$$\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda$$ Constant of separation

$$\begin{cases} X'' + \lambda X = 0 \\ \dot{T} = -\lambda T \iff T = Ce^{-\lambda t} \end{cases}$$

$$\underline{\underline{X}'' + \lambda \underline{X} = 0} \qquad \underline{X}(x) \equiv \underline{X}(x + 2\pi)$$

Periodicity.

To get periodic solutions we must have $\lambda > 0$. Hence

$$X(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

Period $2\pi \iff \sqrt{\lambda} = n \quad (n = 0, \pm 1, \pm 2, \dots)$

We have found [the periodic solutions; no initial values yet.]

$$u_n(x,t) = \left(A_n \cos(nx) + B_n \sin(nx)\right) e^{-n^2 t}$$

<u>Superposition</u>

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) e^{-n^2 t}$$

<u>Initial Values</u>

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

This is the FOURIER SERIES

<u>Remark:</u> All the time

$$\int_0^{2\pi} u(x,t)\,dx = \int_0^{2\pi} f(x)\,dx.$$

HEAT EQN FOR INFINITE THREAD (ROD)

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy \qquad (t>0)$$

See homepage!

Remark: $\lim_{t\to 0+} u(x,t) = f(x)$ **Verification:**

subst. $y - x = z\sqrt{4kt}, \quad t > 0.$

$$dy = \sqrt{4kt}\,dz$$

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \underbrace{f(x+\sqrt{4kt}\,z)}_{\longrightarrow f(x) \text{ as } t\to 0+} dz$$

$$\xrightarrow[t \to 0+]{} f(x) \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-z^2} dz}_{=\sqrt{\pi}} = f(x).$$

Ex $\begin{cases} u(x,0) = e^{-x} & \text{INITIAL TEMPERATURE} \\ u_t = k u_{xx} & (-\infty < x < \infty,\ t > 0) \end{cases}$

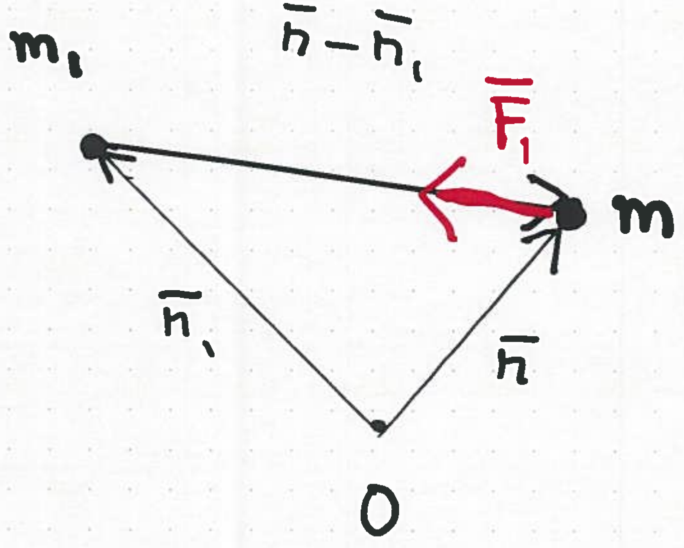
"one of the few fortunate examples"

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} e^{-y} dy$$

$$= \underbrace{\cdots\cdots}_{\text{Tedious.}} = e^{kt - x} \quad \#$$

LAPLACE'S EQN FROM NEWTON'S GRAV. L.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad 1787$$



$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{F}_1 = -\gamma m m_1 \frac{\hat{e}}{|\bar{r} - \bar{r}_1|^2}$$

$$= -\gamma m m_1 \frac{\bar{r} - \bar{r}_1}{|\bar{r} - \bar{r}_1|^3}$$

$$|\bar{r} - \bar{r}_1| = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$$

$$V_1 = V_1(x, y, z) = \frac{-m_1}{|\bar{r} - \bar{r}_1|} = -m_1 \left\{ (x-x_1)^2 + \cdots \right\}^{-\frac{1}{2}}$$

$$\frac{\partial V_1}{\partial x} = m_2 \frac{x - x_1}{|\bar{r} - \bar{r}_1|^3}, \quad \frac{\partial V_1}{\partial y} = m_1 \frac{y - y_1}{|\bar{r} - \bar{r}_1|^3}, \dots$$

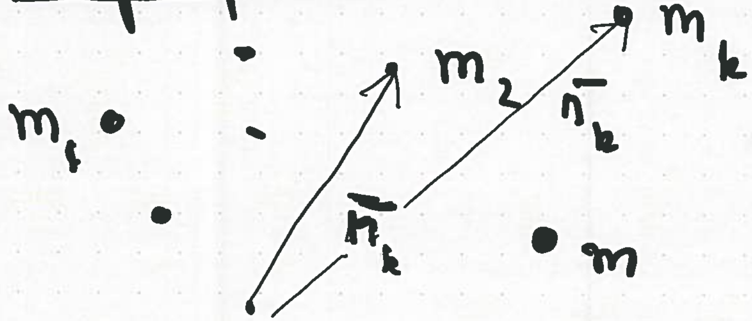
$$\nabla V_1 = m_1 \frac{\bar{r} - \bar{r}_1}{|\bar{r} - \bar{r}_1|^3}$$

$$\nabla^2 V_1 = \nabla \cdot (\nabla V_1) = \dots = 0, \text{ when } \bar{r} \neq \bar{r}_1$$

Thus V_1 satisfies Laplace's eqn.

$$\boxed{\bar{F}_1 = -\gamma m \nabla V_1}$$

Superposition



$$\bar{F} = \sum_j \bar{F}_j$$

$$\bar{F} = -\gamma m \sum_j m_j \frac{\bar{r} - \bar{r}_j}{|\bar{r} - \bar{r}_j|^3} =$$

$$= \gamma m \left\{ \sum \nabla \frac{m_j}{|\bar{r} - \bar{r}_j|} \right\} = -\gamma m \nabla V$$

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$$V = -\sum_j \frac{m_j}{|\bar{r} - \bar{r}_j|}$$

$$\boxed{\begin{aligned} \bar{\bar{F}} &= -\gamma m \nabla V \\ \nabla^2 V &= 0 \quad \text{outside the masses} \end{aligned}}$$

LAPLACE's EQN.

$$V(x,y,z) = \iiint_{\Omega} \frac{\overbrace{\rho \, d\xi \, d\eta \, d\zeta}^{dm}}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}$$

$$\underline{\nabla^2 V = 0}.$$