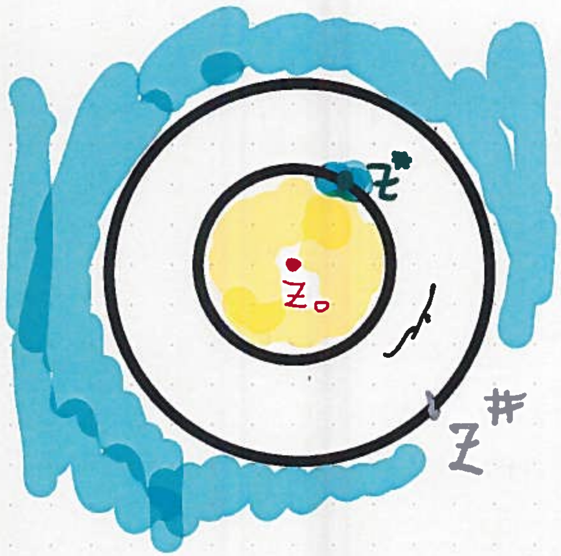
THM (Abel) If the power series $\sum a_n(z-z_0)^n$ converges at the point $z^* \neq z_0$, then the series converges even absolutely, when

$$|z_0 - z| < |z_0 - z^*|.$$

STRICT $<$

If it diverges at the point $z^\# \neq 0$, so it does when $|z_0 - z| > |z_0 - z^\#|$.



CONVERGENCE

DIVERGENCE

__Proof__ Suppose that $\sum a_n(z^* - z_0)^n$ converges, $z^* \neq z_0$. Then __the__ __terms__ __are__ __bounded__:

$$|a_n(z^*-z_0)^n| < M, \quad n = 1, 2, 3, \dots$$

Now take $|z-z_0| < |z^*-z_0|$. Then

$$0 \le |a_n(z-z_0)^n| = \left| a_n(z^*-z_0)^n \left(\frac{z-z_0}{(z^*-z_0)}\right)^n \right|$$

$$= |a_n(z^*-z_0)^n| \left|\frac{z-z_0}{z^*-z_0}\right|^n \le M q^n \quad \text{where now}$$

$$q = \left|\frac{z-z_0}{z^*-z_0}\right| < 1.$$

$\sum M q^n$ converges (GEOM. SERIES) $\xRightarrow[\text{Principle}]{\text{Comparison}}$ $\sum |a_n(z-z_0)^n|$ converges

$\xRightarrow{\text{Thm. *)}}$ $\sum a_n(z-z_0)^n$. $\square$

*) <u>Remark</u> $0 \le b_n \le c_n, \quad n = 1, 2, 3, \dots$

& $\sum c_n < \infty \implies \sum b_n$ converges [Comparison principle]

RADIUS OF CONVERGENCE

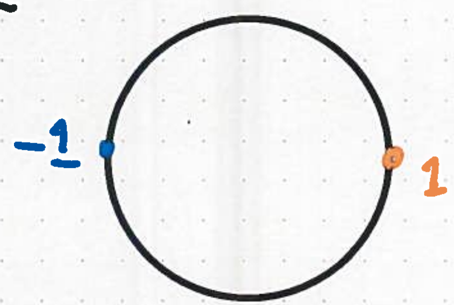
A series

$$\sum a_n (z-z_0)^n$$

has a radius of convergence R, $0 \le R \le \infty$. The series converges (even absolutely), when $|z-z_0| < R$ (Strict) and diverges when $|z-z_0| > R$ (Strict).

<u>Ex:</u> $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $\sum \frac{(-1)^n}{n}$ conv. (Leibniz's test), $\Rightarrow \boxed{R \ge 1.}$

$\sum \frac{(1)^n}{n} = \infty$ (Harmonic series, Integral test) $\Rightarrow \boxed{R \le 1.}$



$R = 1$

FORMULAS FOR R.

When the limits exist ($+\infty, 0$ allowed) we have

$$\boxed{R = \frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}}, \quad \boxed{R = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}}}$$

<u>Ex.:</u> $\sum_{n=1}^{\infty} \frac{z^n}{n^{2020}}, \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \sum_{n=1}^{\infty} \frac{z^n}{n}, \sum_{n=0}^{\infty} z^n,$

$\sum_{n=1}^{\infty} n z^n, \sum_{n=1}^{\infty} n^2 z^n, \dots, \sum_{n=1}^{\infty} n^{2000} z^n$

The formula yields $R = 1$ in all these cases!

<u>Proof</u> of $\frac{1}{R} = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ (provided that the limit exists). Assume $z_0 = 0$ for simplicity.

$$\sqrt[n]{|a_n z^n|} = \sqrt[n]{|a_n|}\,|z| \qquad \left(\sqrt[n]{|a_n|} = \underline{\text{real}} \text{ root}\right)$$

The limit of the n^{th} root of the n^{th} term is strictly less than $\underline{1}$ when :

$$\lim_{n\to\infty}\left(\sqrt[n]{|a_n z^n|}\right) = |z| \lim \sqrt[n]{|a_n|} < 1$$

$$\iff |z| < \frac{1}{\lim_{n\to\infty}\sqrt[n]{|a_n|}}$$

The root test implies the result. $\square$

UNIQUENESS $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n$

when $|z-z_0| < R \implies a_n = b_n, \; n = 0, 1, 2, 3, \ldots$

DIFFERENTIATION. The power series

$$\sum a_n(z-z_0)^n \text{ and } \sum n a_n (z-z_0)^{n-1}$$

have the same radius of convergence, say R.

Furthermore

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$$

TERMWISE DIFFERENTIATED.

when $|z - z_0| < R$. (In general, <u>not</u> $\leq$)

<u>Ex.:</u>

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots \qquad |z| < 1$$

$$\left(\frac{1}{1-z}\right)^2 = 1 + 2z + 3z^2 + 4z^3 + \cdots, \qquad |z| < 1$$

$$\frac{2}{(1-z)^3} = 2 + 2\cdot 3z + 4\cdot 3z^2 + \cdots, \qquad |z| < 1$$

$$\frac{2\cdot 3}{(1-z)^4} = 2\cdot 3 + 4\cdot 3\cdot 2z + \cdots, \qquad |z| < 1$$

$$\vdots$$

Termwise differentiation/integration is valid **strictly** inside the disc of convergence:

$$|z - z_0| < R.$$

MULTIPLICATION (Cauchy's Rule)

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = a_0 b_0 + (a_1 b_0 + a_0 b_1) z$$

$$+ (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) z^3$$

(at least when)

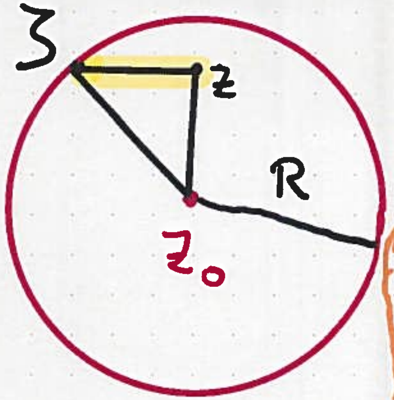
$+ \dots$ when $|z| < \min\{R_a, R_b\}$

TAYLOR EXPANSION

Suppose that $f(z)$ is analytic in the domain Ω and that the disc $|z - z_0| \leq R$ belongs to Ω. Then

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \text{when } |z - z_0| < R}$$

PROOF.



$|\zeta - z_0| = R$

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

<u>Recall</u>:

- $$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta .$$

- $$1 + q + q^2 + \cdots + q^{n-1} + \frac{q^n}{1-q} = \frac{1}{1-q}, \quad |q| < 1 .$$

$$q = \frac{z - z_0}{\zeta - z_0}$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 + z_0 - z} = \frac{1}{\zeta - z_0}\left[\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}\right]$$

$$= \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \cdots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n}$$

$$+ \frac{(z - z_0)^n}{(\zeta - z_0)^n (\zeta - z)}$$

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(\zeta)\,d\zeta}{\zeta - z_0} + \frac{1}{2\pi i}\oint \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^2}\cdot \underline{(z-z_0)}$$

$$+ \cdots + \frac{1}{2\pi i}\oint \frac{f(\zeta)\,d\zeta}{(\zeta - z)^n}\cdot (z-z_0)^{n-1}$$

$$+ \frac{1}{2\pi i}\oint \frac{f(\zeta)}{\zeta - z}\left(\frac{(z-z_0)}{(\zeta - z_0)}\right)^n d\zeta$$

$$= f(z_0) + f'(z_0)\underline{(z-z_0)} + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1}$$

$$+ \underbrace{\frac{1}{2\pi i}\oint \frac{f(\zeta)}{(\zeta - z)}\left(\frac{z-z_0}{\zeta - z_0}\right)^n d\zeta}_{R_n} \qquad (|\zeta - z_0| = R)$$

$|\zeta - z| \geq R - |z - z_0|$

$$|R_n| \leq \frac{1}{2\pi}\max_{\bigcirc}|f|\,\underbrace{\frac{1}{R-|z-z_0|}}_{>0}\underbrace{\left|\frac{z-z_0}{R}\right|^n}_{<1}$$

The remainder approaches zero

$\longrightarrow 0$ as $n \to \infty$. Hence

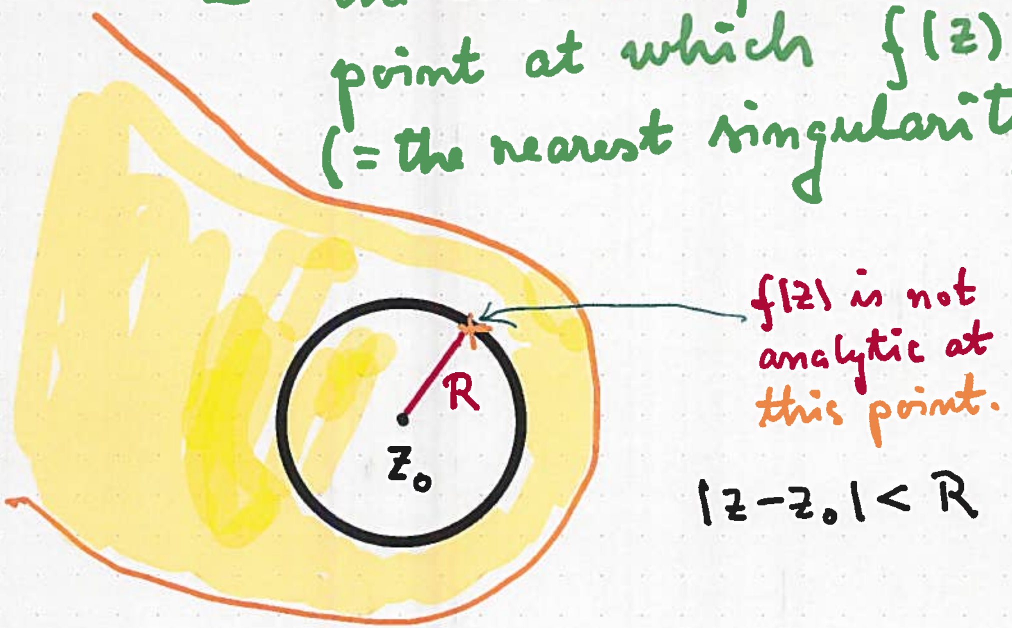
$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots\cdots$$

$\square$

whe $|z-z_0| < R$.

PRACTICAL RULE: $R = \text{distance}(z_0, \partial\Omega)$

= the distance from the center z_0 to the nearest point at which $f(z)$ is no longer analytic (= the nearest singularity).



The largest disc, centered at z_0, in which $f(z)$ is analytic.