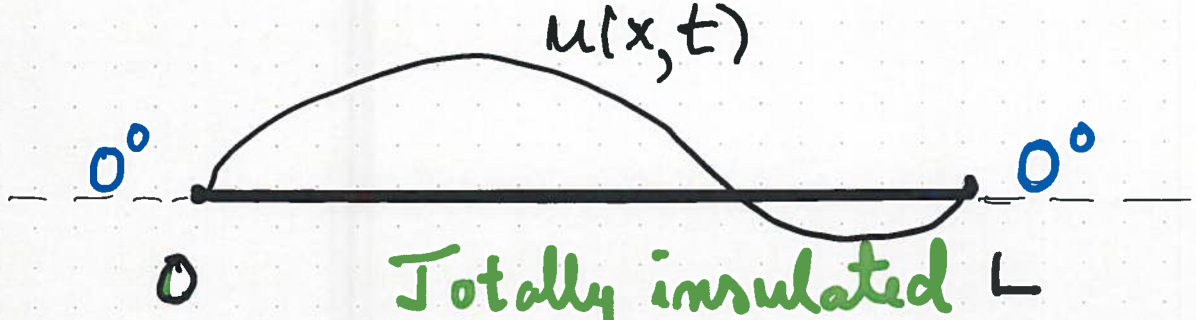
ONE DIMENSIONAL HEAT EQN



Totally insulated rod (thread)

Heat Eqn.

$$\boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad k > 0}$$

$u = u(x,t)$
$= \text{temperature}$

$$\begin{cases} u(0,t) = 0 = u(L,t) \\ u(x,0) = f(x) \end{cases}$$

endpoints kept at temp. 0° Celcius.

INITIAL TEMPERATURE

SEPARATION OF VARIABLES.

I SEPARATION

$$u(x,t) = X(x)\,T(t) \qquad \left(X(x) = ? \quad T(t) = ? \right)$$

$$\underbrace{X\dot{T}}_{u_t} = kX''T \qquad \dot{T} = \frac{dT}{dt}$$

$$\underbrace{\frac{X''}{X}}_{\text{Does not depend on } t} = \underbrace{\frac{1}{k}\frac{\dot{T}}{T}}_{\text{Does not depend on } x} = \lambda \text{ (constant of separation)}$$

$$\begin{cases} X'' = \lambda X, & \underline{X(0) = X(L) = 0} \text{ Endpoints.} \\ \dot{T} = \lambda k T \end{cases}$$

II The X-eqn.

1°) $\underline{\lambda > 0}$. No solution $X(x)$ can satisfy the endpoint conditions, except $X(x) \equiv 0$

Please, verify!

2°) $\underline{\lambda = 0}$. Again $X(x) \equiv 0$.

3°) $\underline{\lambda < 0}$, say $\lambda = -\omega^2$, $\omega \neq 0$.

$$X''(x) + \omega^2 X(x) = 0 \iff$$

$$X(x) = \cancel{A\cos(\omega x)} + B\sin(\omega x)$$

$$0 = X(0) = \underline{A} \qquad X(x) = B\sin(\omega x)$$

$$0 = X(L) = B\sin(\omega L) \iff$$

$$\omega L = n\pi \quad \text{or} \quad B = 0.$$

$$\omega = \frac{n\pi}{L}, \quad n = 0, \pm 1, \pm 2, \dots$$

- $\underline{T\text{-eqn.}}$ $\quad \dot{T} = k\lambda T = -k\omega^2 T$

$$= -k\left(\frac{n\pi}{L}\right)^2 T,$$

$$T(t) = C e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$\Longrightarrow \quad u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-\frac{k n^2 \pi^2}{L^2} t}$$

$$(n = 0, \pm 1, \pm 2, \ldots)$$

III SUPERPOSITION: $u(x,t) = \sum_{n=-\infty}^{\infty} u_n(x,t)$.

- $$u(x,t) = \sum_{\cancel{n=-\infty}\; n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2 \frac{k\pi^2 t}{L^2}}$$

See below!

$(-n)^2 = n^2$

We skip $n = -1, -2, -3, \ldots$

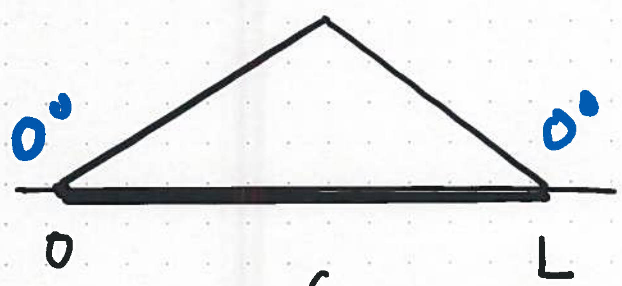
IV INITIAL TEMPERATURE

$$f(x) \overset{?}{=} u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cdot 1$$

FOURIER SINE SERIES

- $$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Ex.



$$f(x) = \begin{cases} x, & 0 \le x \le \frac{L}{2} \\ L-x, & \frac{L}{2} \le x \le L \end{cases}$$

$$B_n = \begin{cases} 0, & n = 2, 4, 6, 8, \ldots \\ \frac{4L}{n^2\pi^2}, & n = 1, 5, 9, \ldots \\ -\frac{4L}{n^2\pi^2}, & n = 3, 7, 11, \ldots \end{cases}$$

Answer: $$u(x,t) = \frac{4L}{\pi^2}\left\{ e^{-\frac{\pi^2 k t}{L^2}} \sin\left(\frac{\pi x}{L}\right) - \frac{1}{9} e^{-\frac{9\pi^2 k t}{L^2}} \sin\left(\frac{3\pi x}{L}\right) + \cdots \right\}$$

Remark: $$\lim_{t\to\infty} u(x,t) = 0. \qquad \#$$

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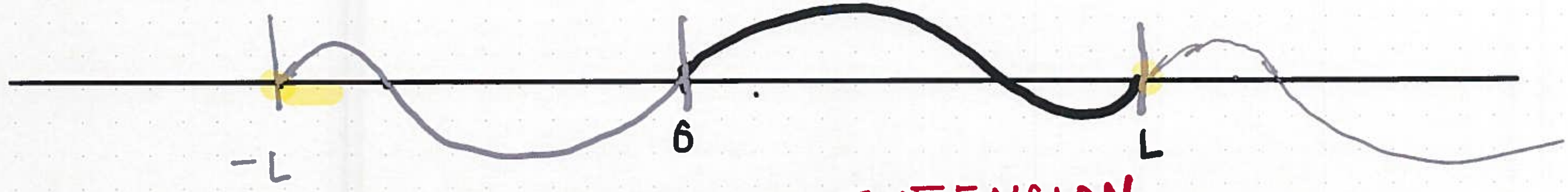
d'Alembert's Solution of the Wave Eqn.

Jean-Baptiste le Rond d'Alembert 1717–1783.

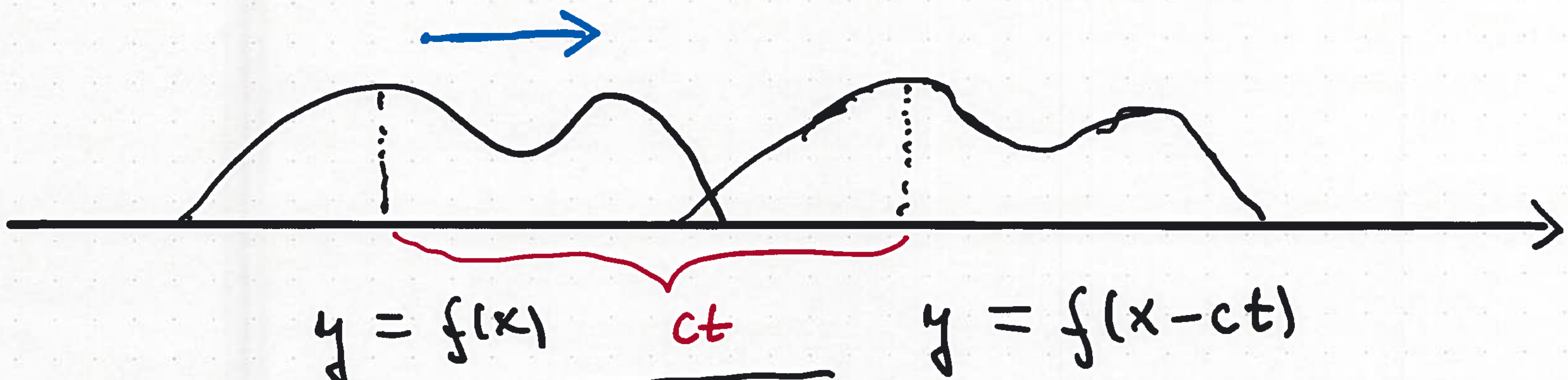
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & u = u(x,t) \\ & \underline{-\infty < x < \infty} \; ! \\ u(x,0) = f(x) & \text{INITIAL SHAPE} \\ u_t(x,0) = g(x) & \text{—"— SPEED} \end{cases}$$



ODD PERIODIC EXTENSION

Period $= 2L$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$



$\boxed{c = \text{speed!!}}$

Remark

$$\frac{\partial}{\partial x}\int_{a(x,t)}^{b(x,t)} g(\xi)\,d\xi = g(b(x,t))\frac{\partial b(x,t)}{\partial x} - g(a(x,t))\frac{\partial a(x,t)}{\partial x}$$

LEIBNIZ's RULE

Ex: $$\frac{\partial}{\partial t}\int_{x+7t}^{t^2} e^{-\xi^2}\,d\xi = e^{-(t^2)^2}\cdot 2t - e^{-(x+7t)}\cdot 7$$

Compare: $$\frac{d}{dx}\int_a^x g(y)\,dy = g(x)$$

$$\frac{d}{dx}\int_x^b g(y)\,dy = -g(x)$$

How to find the formula?

1°) The general solution of $u_{tt} = c^2 u_{xx}$ is

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

This comes by changing variables:

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases} \qquad u(x,y) = \bar{u}(\xi, \eta)$$

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \overset{!}{=} -4c^2 \frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}$$

Calculate using Chain Rule.

$$\frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} = 0 \iff \frac{\partial \bar{u}}{\partial \xi} = F(\xi)$$

$$\iff \bar{u} = \underbrace{\int F(\xi) d\xi}_{\phi(\xi)} + \underbrace{G(\eta)}_{\psi(\eta)}$$

2°) INITIAL COND.

$$f(x) = u(x,0) = \phi(x) + \psi(x) \qquad (\text{I})$$

$$g(x) = u_t(x,0) = c[\phi'(x) - \psi'(x)] \qquad (\text{II})$$

Integrate II from $x - ct$ to $x + ct$

$$\frac{1}{c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi = \phi(x+ct) - \phi(x-ct) + \psi(x-ct) - \psi(x+ct) \qquad (\mathrm{III})$$

Replace x by $x \pm ct$ in (I). We get

$$\begin{cases} f(x+ct) = \phi(x+ct) + \psi(x+ct) & (\mathrm{I}_+) \\ f(x-ct) = \phi(x-ct) + \psi(x-ct) & (\mathrm{I}_-) \end{cases}$$

Adding up like "$\mathrm{III} + \mathrm{I}_+ + \mathrm{I}_-$" yields us

$$\frac{1}{c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi + f(x+ct) + f(x-ct) =$$

$$2\phi(x+ct) + 2\psi(x-ct) = 2u(x,t)$$

This IS d'Alembert's formula.

ADVANTAGE: speed c and shape visible.