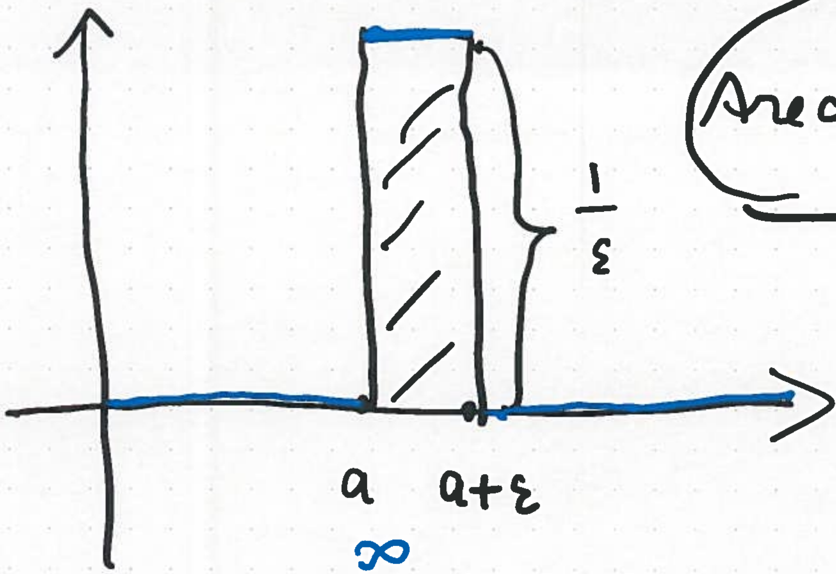


Area = 1

$\delta_\varepsilon(t-a) =$ PICTURE

SUDDEN IMPULSE!
centered at $t = a$

$$\lim_{\varepsilon \to 0^+} \int_0^\infty \delta_\varepsilon(t-a) f(t)\,dt = f(a)$$

$$\boxed{\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)}$$

<u>Proof</u>: $$\int_0^\infty \delta_\varepsilon(t-a) f(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon} f(t)\,dt$$

$$\left| f(a) - \frac{1}{\varepsilon}\int_a^{a+\varepsilon} f(t)\,dt \right| = \left| \frac{1}{\varepsilon}\int_a^{a+\varepsilon} (f(a) - f(t))\,dt \right|$$

$$\leq \frac{1}{\varepsilon}\int_a^{a+\varepsilon} |f(a)-f(t)|\,dt \leq \frac{1}{\cancel{\varepsilon}} \cdot \cancel{\varepsilon} \cdot \max_{a \leq t \leq a+\varepsilon} |f(a)-f(t)|$$

$\longrightarrow 0$ as $\varepsilon \to 0+$ by continuity

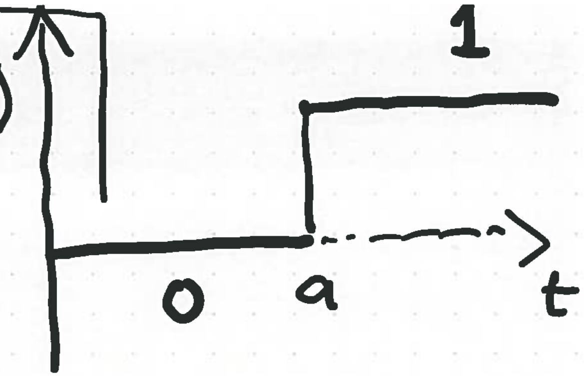
($f(t)$ is continuous here). $\square$

RULES

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}}$$

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$$

$$\boxed{\frac{d\,u(t-a)}{dt} = \delta(t-a)}$$

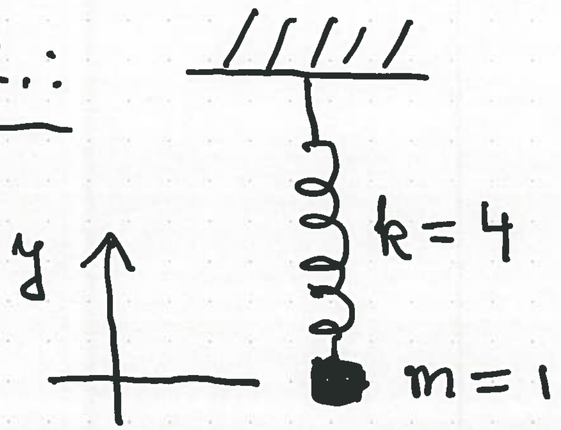


$$\delta_\varepsilon(t-a) = \frac{u(t-a) - u(t-a+\varepsilon)}{\varepsilon} \longrightarrow \delta(t-a)$$

Difference quotient.

Formally $\boxed{\frac{d}{dt}\,\underbrace{u(t-a)}_{\text{Heaviside}} = \underbrace{\delta(t-a)}_{\text{Dirac's delta}}}$

Ex.:



Hammerblow at $t=\pi$

$$\begin{cases} y''(t) + 4y(t) = \delta(t-\pi) \\ y(0) = 0,\ y'(0) = 0 \end{cases}$$

[Sudden impulse at $t=\pi$]

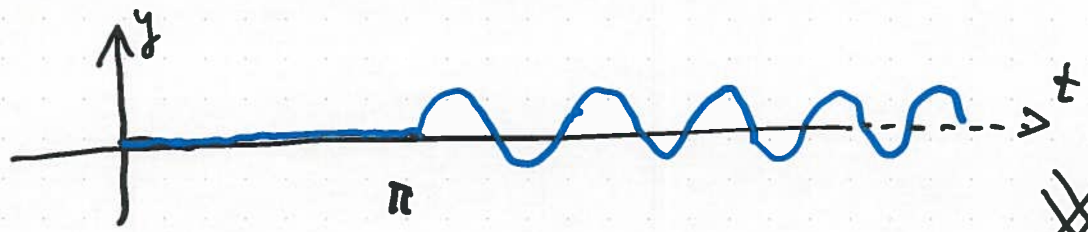
$$(s^2+4)\,Y(s) = e^{-\pi s}$$

$$Y(s) = \frac{e^{-\pi s}}{s^2+4}$$

$$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$$

$$y(t) = u(t-\pi)\sin[2(t-\pi)]\cdot\frac{1}{2}$$

Answer: $y(t) = \frac{1}{2}u(t-\pi)\sin(2t)$



CONVOLUTION

DEF. $$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau .$$

$$\boxed{\mathcal{L}\{\underbrace{(f*g)(t)}_{\text{CONVOL.}}\} = \underbrace{F(s)\cdot G(s)}_{\text{PRODUCT}}}$$

(**Remark:** In Fourier analysis, signal analysis, etc. one uses
$$(f*g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy .)$$
Ignore now! Comes later!

Ex: $f(t) = e^t$, $g(t) = e^{-t}$ $\quad [\, f(t) g(t) = 1 \,]$

$$(f*g)(t) = \int_0^t e^{\tau} e^{-(t-\tau)}\, d\tau = \int_0^t e^{2\tau - t}\, d\tau$$

t

$$= e^{-t}\frac{1}{2}\Big|_0 e^{2\tau} = \frac{1}{2}e^{-t}(e^{2t}-1) = \sinh(t).$$

<u>Rules:</u> $\quad f * g = g * f \quad$ COMMUTATIVE

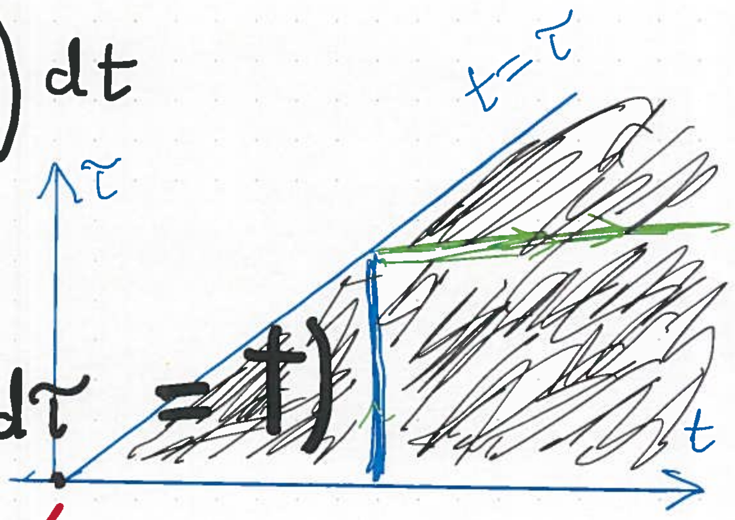
$$(f*g)*h = f*(g*h) \quad \text{ASSOCIATIVE}$$

$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$, see above.

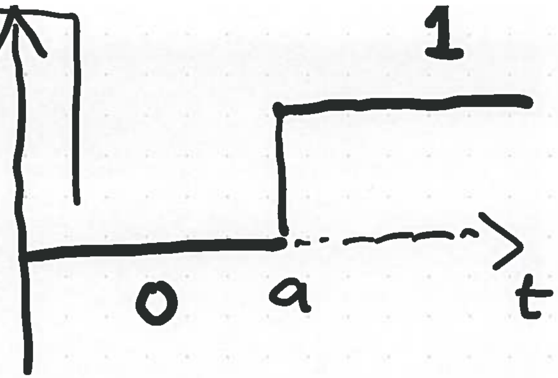
<u>Proof</u> $\quad \mathcal{L}\{(f*g)(t)\} \overset{\text{DEF}}{\underset{\text{of } \mathcal{L}}{=}} \int_0^\infty (f*g)(t)\, e^{-st}\, dt$

$$\underset{\text{Def of } *}{=} \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right) dt$$

$$= \int_0^\infty f(\tau)\left(\underbrace{\int^\infty}_{t=\tau} e^{-st} g(t-\tau)\, dt\right) d\tau$$



$$\boxed{\frac{d\,u(t-a)}{dt} = \delta(t-a)}$$



$$\delta_\varepsilon(t-a) = \frac{u(t-a) - u(t-a+\varepsilon)}{\varepsilon} \longrightarrow \delta(t-a)$$

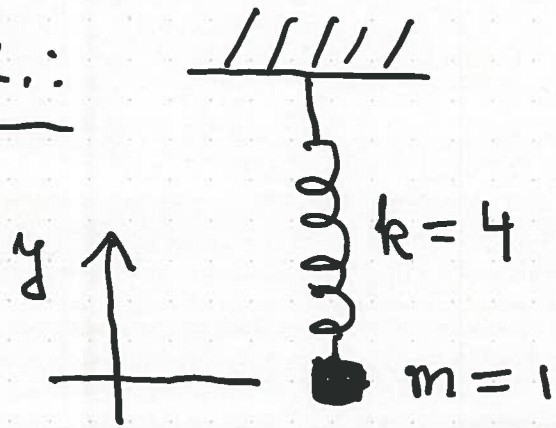
Difference quotient.

Formally $\boxed{\frac{d}{dt} u(t-a) = \delta(t-a)}$

Heaviside   Dirac's delta

Hammerblow at $t=\pi$

Ex.:



$$\begin{cases} y''(t) + 4y(t) = \delta(t-\pi) \\ y(0)=0,\ y'(0)=0 \end{cases}$$

[Sudden impulse at $t=\pi$]

Intermediate Integral

$$\spadesuit = \int_{\tau}^{\infty} e^{-st} g(t-\tau)\,dt = \int_0^{\infty} e^{-s(\tau+T)} g(T)\,dT$$

$$\left.\begin{aligned} t-\tau &= T \\ dt &= dT \end{aligned}\right| = e^{-s\tau}\int_0^{\infty} e^{-sT} g(T)\,dT$$

$$= \underline{e^{-s\tau} G(s)}$$

$$\dagger) = \int_0^{\infty} f(\tau)\, \underline{e^{-s\tau} G(s)}\, d\tau$$

$$= G(s)\int_0^{\infty} f(\tau) e^{-s\tau}\, d\tau = G(s)F(s) \quad \blacksquare$$

INTEGRAL EQNS

CONVOLUTION !

$$\begin{cases} y(t) = e^{-t} + 2\int_0^t e^{-3\tau}\, y(t-\tau)\, d\tau \\ y(t) = ? \end{cases}$$

(VOLTERRA)

[In passing, notice $y(0) = 1$]

$$Y(s) = \frac{1}{s+1} + 2\,\mathcal{L}\{e^{-3t}\}\,Y(s)$$

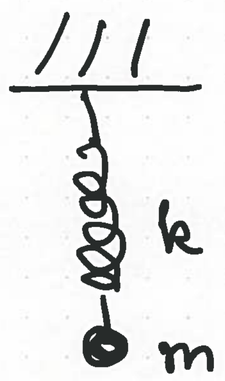
$$= \frac{1}{s+2} + 2\,\frac{1}{s+3}\,Y(s)$$

$$Y(s) = \frac{s+3}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

$$y(t) = e^{-t} + 2te^{-t}.$$
[Answer]

s-shift.

MASS-SPRING SYSTEM/RESONANCE

Ex. 

$$my'' + ky = A\sin(\omega t)$$

$$y'' + \frac{k}{m}y = \frac{A}{m}\sin(\omega t)$$

$$\left(\omega_0^2 = \frac{k}{m}\right) \qquad y'' + \omega_0^2 y = B\sin(\omega t)$$

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$s^2 Y(s) + \omega_0^2 Y(s) = \frac{B\omega}{s^2+\omega^2}$$

$$Y(s) = B\omega \frac{1}{s^2+\omega^2} \cdot \frac{1}{s^2+\omega_0^2}$$

convolution:

$$y(t) = B\omega \int_0^t \frac{\sin(\omega\tau)}{\omega} \frac{\sin(\omega_0(t-\tau))}{\omega_0}\, d\tau$$

$$2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$$

$$y(t) = \begin{cases} \dfrac{B}{2\omega_0}\left[\dfrac{\sin(\omega_0 t) - \sin(\omega t)}{\omega - \omega_0} + \dfrac{\sin(\omega_0 t) + \sin(\omega t)}{\omega + \omega_0}\right], & \omega \neq \omega_0 \\ \dfrac{B}{\omega}\left[\dfrac{1}{2\omega}\sin(\omega t) - \dfrac{t}{2}\cos(\omega t)\right] & \omega = \omega_0 \end{cases}$$

RESONANCE

UNBOUNDED!

$t \to \infty$

Better?

$$\frac{B\omega}{\omega^2 - \omega_0^2}\left[\frac{1}{s^2+\omega_0^2} - \frac{1}{s^2+\omega^2}\right].$$

$\square$

Ex. *) $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$ via Laplace transform.

(* means ADVANCED Ex.)

$$\boxed{\int_0^\infty \left|\frac{\sin x}{x}\right| dx = +\infty}$$

$$f(t) = \int_0^\infty \frac{\sin(tx)}{x}\,dx \qquad (t > 0)$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} \int_0^\infty \frac{\sin(tx)}{x}\,dx\,dt$$

$$= \int_0^\infty \frac{1}{x}\Bigg(\underbrace{\int_0^\infty e^{-st}\sin(tx)\,dt}_{=\frac{x}{s^2+x^2}}\Bigg)dx$$

Integrations by parts.

$$= \int_0^\infty \frac{dx}{s^2+x^2} = \pi/2s$$

The inverse is

$$f(t) = \frac{\pi}{2}$$

In fact

$$\int_0^\infty \frac{\sin(ax)}{x}\,dx = \begin{cases} \frac{\pi}{2}, & a > 0 \\ 0, & a = 0 \\ -\frac{\pi}{2}, & a < 0 \end{cases}$$

[Subst. $ax = y$]

Another important integral is

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$