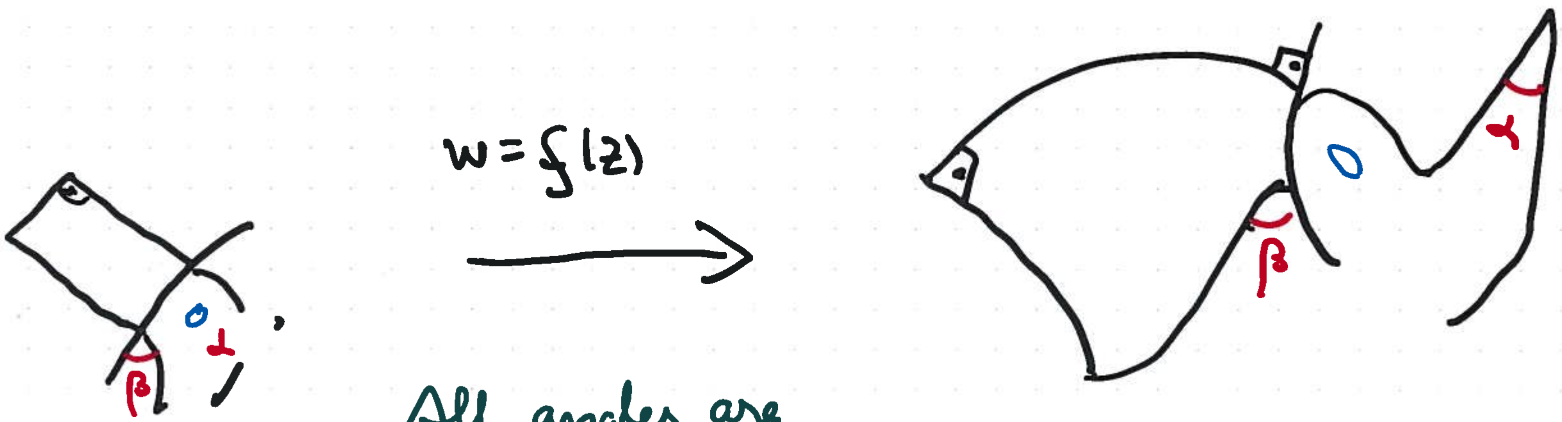
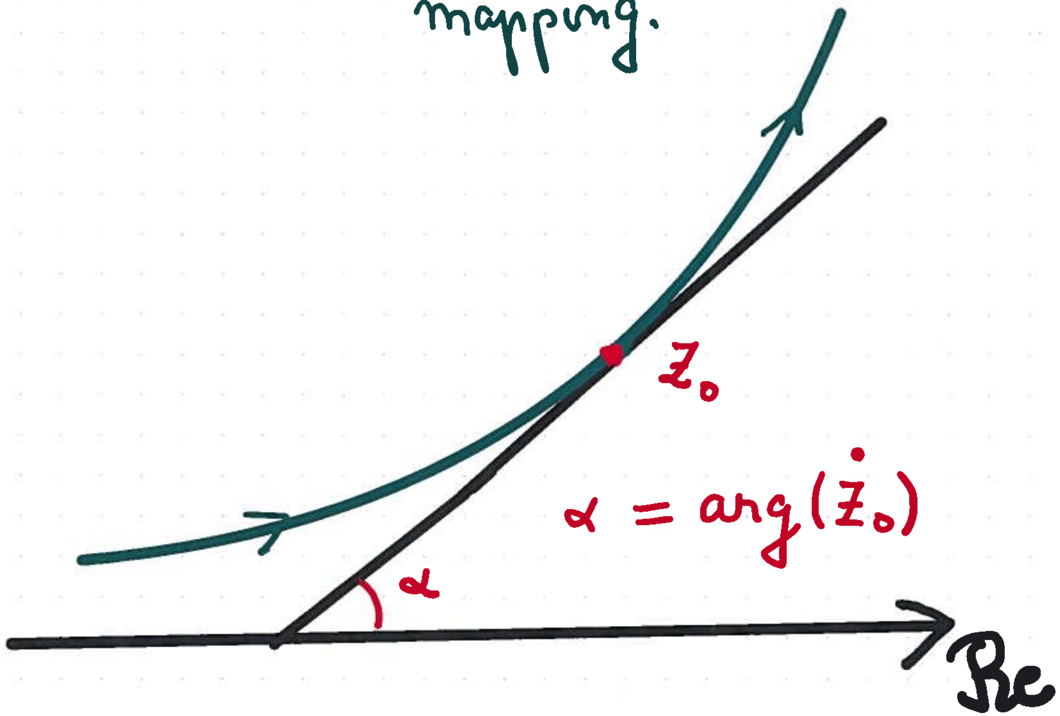
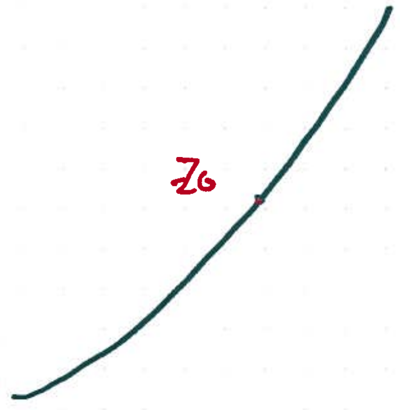



All angles are preserved under a conformal mapping.




$$z = z(t)$$
$$z_0 = z(t_0)$$
$$\dot{z} = \dot{z}(t)$$
$$\dot{z}_0 = \dot{z}(t_0)$$



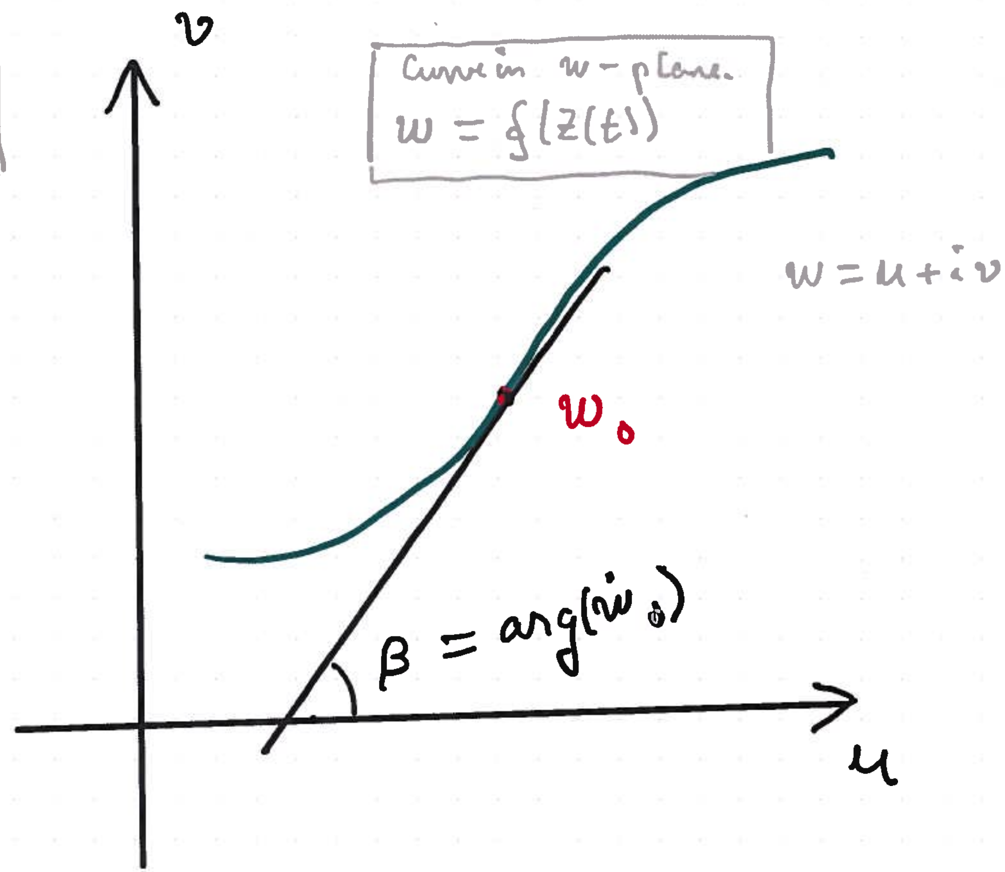

$z = z(t$

Curve in z-plane

$$\dot{z}_0 = \dot{z}(t_0)$$

$$w = f(z(t))$$

$$w_0 = f(z(t_0)) = f(z_0)$$



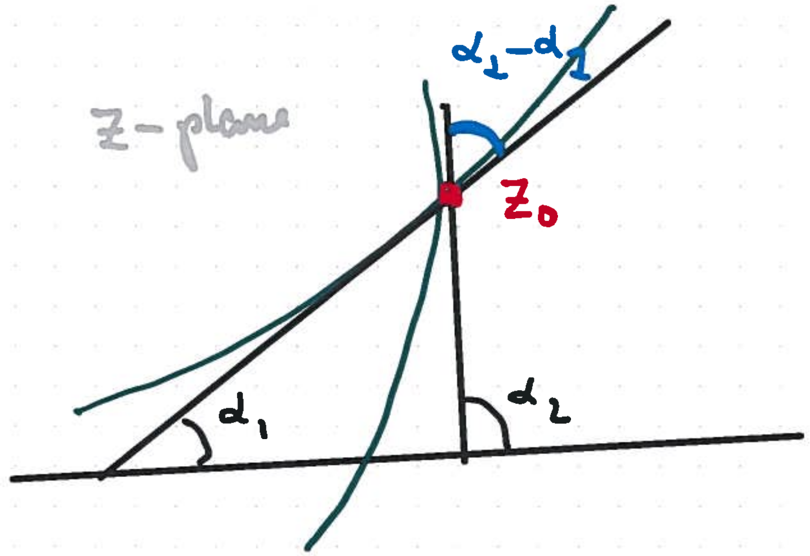

$$\dot{w} = \frac{dw}{dt} = f'(z(t))\frac{dz}{dt}$$

$$\dot{w}(t_0) = f'(z_0)\,\dot{z}_0$$

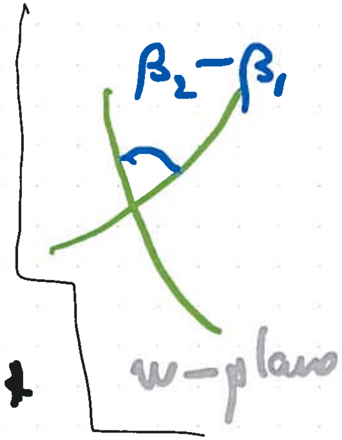
$$\dot{w}_0 = \dot{w}(t_0)$$

$$\boxed{\beta} = \arg(f'(z_0)\,\dot{z}_0) = \arg(f'(z_0)) + \arg(\dot{z}_0)$$

$$\boxed{= \arg(f'(z_0)) + \alpha}$$




$$\underline{\beta_2-\beta_1} = \cancel{\arg(f'(z_0))} + \alpha_2 - \cancel{\arg(f'(z_0))} - \alpha_1 = \underline{\alpha_2-\alpha_1}$$

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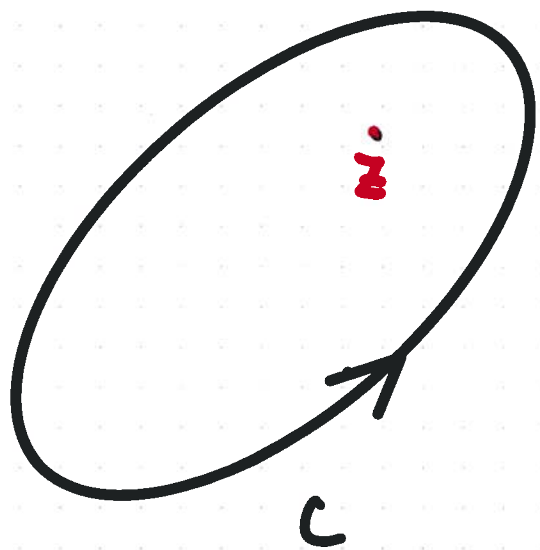
$$\boxed{\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)}$$

CAUCHY



## ALGEBRA

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

$$= a_n (z-z_1)(z-z_2)\cdots(z-z_n), \quad a_n \neq 0$$

**THM** (Fundamental Thm of Algebra). Every polynomial of degree $n$ has at least one complex root.

**COR** It has $n$ roots (counted according to multiplicity). [Divide them out, one at a time.]

**Preparation** $P(z) = z^n \left( a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n} \right)$

$$|P(z)| = |z|^n \left| \text{—}\,\text{''}\,\text{—} \right|$$

$$\geq |z|^n\Bigg(|a_n| - \underbrace{\left|\frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right|}_{\to 0 \text{ as } z \to \infty}\Bigg)$$

$$\geq |z|^n\left(|a_n| - \frac{|a_n|}{2}\right) = \frac{1}{2}|a_n||z^n|$$

when $|z| \geq R_0$ (= some number)

<u>Proof:</u> <u>Antithesis:</u> $P(z) \neq 0$, for all $z$.

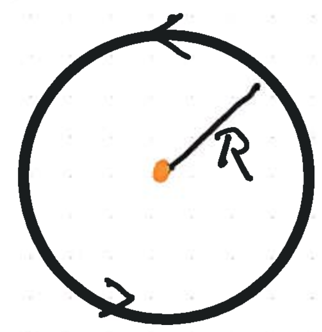
of the Thm

$\underset{\text{ANTITH.}}{\Longrightarrow}$ $f(z) = \dfrac{1}{P(z)}$ is analytic in the whole plane.

$$\frac{1}{P(0)} = \frac{1}{2\pi i}\oint_{|z|=R} \frac{dz}{z\,P(z)}$$

[Cauchy]

$R > R_0$



$$\left|\frac{1}{P(0)}\right| \overset{\text{ML-Ineq.}}{\leq} \frac{1}{2\pi} \oint_{|z|=R} \frac{|dz|}{|z|\,|P(z)|} \leq \frac{1}{2\pi}\cdot 2\pi R \cdot \frac{1}{R\cdot \frac{1}{2}|a_n| R^n}$$

$\underline{R > R_0}$

$$= \frac{2}{|a_n| R^n} \longrightarrow 0 \quad \text{as } R \to \infty.$$

$$\left|\frac{1}{P(0)}\right| = 0 \qquad \lightning$$

Hence the antithesis was false. $\square$

<u>Remark</u>: An other proof goes via Liouville's Theorem.

# COMPLEX SERIES

$$z_1 + z_2 + z_3 + \cdots + z_n + \cdots = \sum_{n=1}^{\infty} z_n$$

$$S_1 = z_1$$
$$S_2 = z_1 + z_2$$
$$S_3 = z_1 + z_2 + z_3$$
$$\vdots$$

**DEF.** $S = \lim_{N\to\infty} S_N$

$$\iff S = \sum_{n=1}^{\infty} z_n$$

The series is convergent if $S$ exists and is finite.

It is divergent in all other cases.

**LEMMA** $\sum z_n$ conv. $\implies \lim_{n\to\infty} z_n = 0$

Necessary (but not sufficient) condition.

Proof: $z_n = S_n - S_{n-1} \longrightarrow S - S = 0$ $\square$

$z_n = x_n + i y_n$ The series splits in real and imaginary parts

$$\sum_{n=1}^{N} z_n = \sum_{n=1}^{N} x_n + i \sum_{n=1}^{N} y_n$$

THM $\sum z_n$ conv. $\Longleftrightarrow$ Both $\sum x_n$ and $\sum y_n$ converge

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

# GEOMETRIC SERIES.

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z} \qquad (z \neq 1)$$

A disc!

$$1 + z + z^2 + \cdots + z^{n-1} + \cdots = \frac{1}{1-z}, \text{ when } |z| < 1$$

The series diverges when $|z| \geq 1$, since the $n^{th}$ term does not go to zero as $n \to \infty$.


$e^{-i\theta} = \cos\theta - i\sin\theta$

$$z = re^{i\theta}$$

$$1 + re^{i\theta} + r^2 e^{2i\theta} + \cdots = \frac{1}{1 - re^{i\theta}} \qquad (0 \leq r < 1)$$

$$1 + r\cos(\theta) + r^2\cos(2\theta) + \cdots = \mathrm{Re}\left\{\frac{1}{1 - re^{i\theta}}\right\}$$

- $$\frac{1}{1 - re^{i\theta}} = \frac{1 - re^{-i\theta}}{(1 - re^{i\theta})(1 - re^{-i\theta})} = \frac{1 - r\cos\theta + r\sin\theta \cdot i}{1 - 2r\cos\theta + r^2}$$

$$1 + r\cos(\theta) + r^2\cos(2\theta) + \cdots = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$$

$$\boxed{\frac{1}{2} + r\cos(\theta) + r^2\cos(2\theta) + \cdots = \frac{1}{2}\,\frac{1 - r^2}{1 - 2r\cos\theta + r^2}}$$ #

THE POISSON KERNEL

**THM** $\sum_{n=1}^{\infty} |z_n| < \infty \Rightarrow \sum_{n=1}^{\infty} z_n$ converges

"Absolutely convergent".

"Conditionally convergent".

<u>Proof</u> $0 \le |x_n| \le |z_n|$, $0 \le |y_n| \le |z_n|$

$\sum |z_n| < \infty \Rightarrow \sum |x_n| < \infty \Rightarrow \sum x_n$ converges

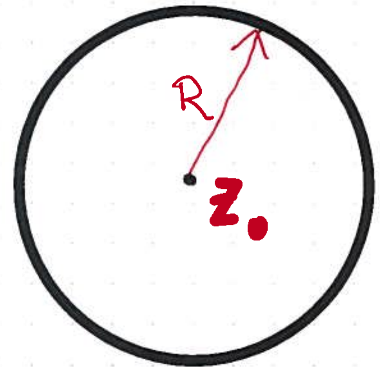
The same for $\sum y_n$. Now $\sum z_n = \sum x_n + i \sum y_n$ $\square$

# POWER SERIES

$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_{n-1}(z-z_0)^{n-1} + \cdots = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Always convergence at $z = z_0$.

- $\sum_{n=0}^{\infty} n!\, z^n$ converges only at $z = z_0$ $R=0$
- $\sum z^n$ conv. $\iff |z| < 1$ $R=1$
- $\sum \frac{z^n}{n!}$ conv. for all $z$ $R=\infty$

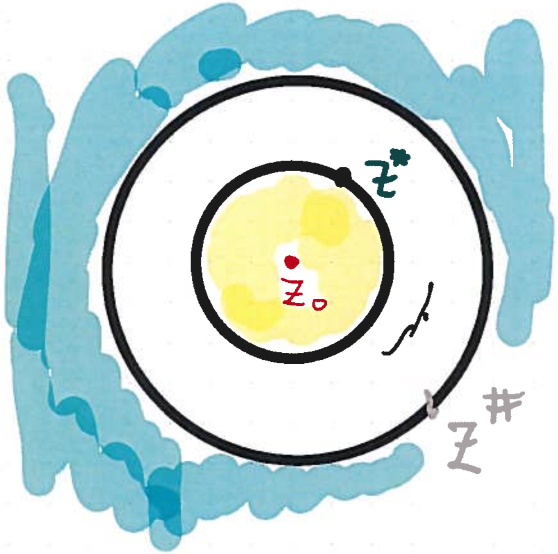


THM (Abel) If the power series $\sum a_n (z-z_0)^n$ converges at the point $z^* \neq z_0$, then the series converges even absolutely, when

$$|z_0 - z| < |z_0 - z^*|.$$

STRICT $<$

If it diverges at the point $z^\# \neq 0$, so it does when $|z_0 - z| > |z_0 - z^\#|$.




CONVERGENCE

~~DIVERGENCE~~

Proof $\quad \sum |a_n (z-z_0)^n| = \sum a_n \frac{z-z_0}{\cdot}$