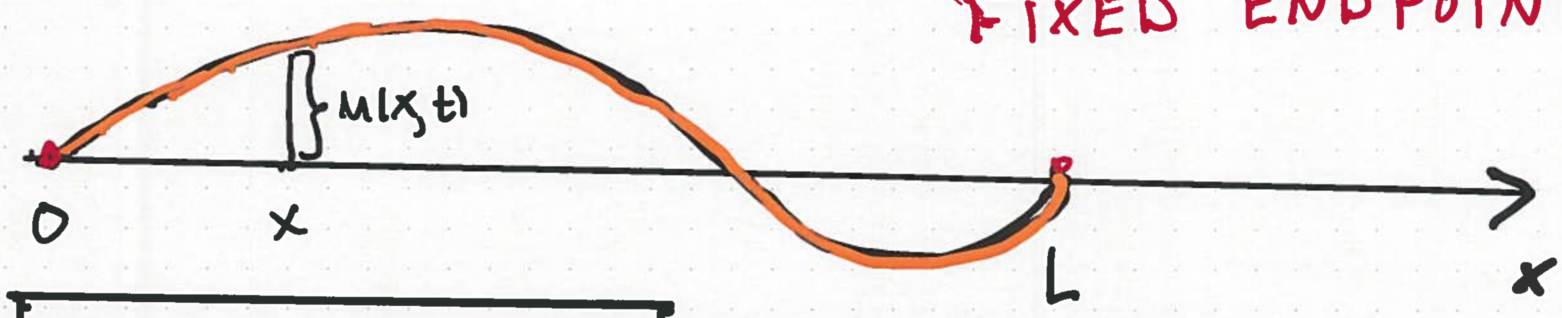
SEPARATION OF VARIABLES

VIBRATING STRING

$u(0,t) = 0 = u(L,t)$

FIXED ENDPOINTS



$$\boxed{u_{tt} = c^2 u_{xx}}$$

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

INITIAL SHAPE

INITIAL SPEED

I <u>Separation</u>

$$u(x,t) = X(x)\,T(t)$$

$$u_{tt} = X \frac{d^2T}{dt^2}$$

$$= X\ddot{T}$$

$$X\ddot{T} = c^2 X'' T$$

$$\underbrace{\frac{X''}{X}}_{\text{Independent of } t} = \underbrace{\frac{1}{c^2}\frac{\ddot{T}}{T}}_{\text{Independent of } x} = -\lambda$$

A CONSTANT!

END POINTS.

$$\begin{cases} X'' + \lambda X = 0; \\ \ddot{T} + \lambda c^2 T = 0 \end{cases} \qquad \underline{X(0) = X(L) = 0}$$

II.

$$\boxed{X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0}$$

1°) $\underline{\lambda = 0}$ $\quad X(x) = ax + b$; $a = 0$, $b = 0$

$X(x) \equiv 0$! "useless"

2°) $\underline{\lambda < 0}$ $\quad X(x) = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$.

VERIFY, please. $\quad \overset{\text{END}}{\underset{\text{POINTS}}{\Longrightarrow}} X(x) \equiv 0$ "useless"

3°) $\underline{\lambda > 0}$ $\quad X(x) = \cancel{a\cos(\sqrt{\lambda}x)} + b\sin(\sqrt{\lambda}x)$

$$X(0) = 0 \iff \underline{a = 0}$$

$$X(L) = 0 \iff b\sin(\sqrt{\lambda}L) = 0$$

$$\iff \sqrt{\lambda}L = \pm n\pi \text{ or } b = 0$$

$$\pm\sqrt{\lambda} = \frac{n\pi}{L} \qquad n = 0, \pm 1, \pm 2, \dots$$

Thus we have

$$X_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right)$$

The corresponding $T(t)$ is then

$$\underline{\underline{\ddot{T} + \lambda c^2 T = 0}}, \quad \pm\sqrt{\lambda} = \frac{n\pi}{L}$$

$$T_n(t) = A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)$$

We have the solutions

$$\left[A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)\right] \cdot \sin\left(\frac{n\pi}{L} x\right)$$

$$n = 0, \pm 1, \pm 2, \ldots$$

They satisfy the endpoint conditions. (Not initial cond., yet!)

III SUPERPOSITION

Add all these solutions and adjust the constants A_n, B_n.

$$u(x,t) = \sum_{\cancel{n=-\infty}\; n=1}^{+\infty} \left[A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\right] \sin\left(\frac{n\pi}{L}x\right)$$

The values $n = -1, -2, -3, \ldots$ are absorbed in the constants.

$\overline{\underline{IV}}$ INITIAL CONDITIONS

$$f(x) \stackrel{?}{=} u(x,0) = \sum_{n=1}^{\infty} A_n \, 1 \, \sin\left(\frac{n\pi}{L}x\right)$$

FOURIER SINE SERIES !

- $$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$g(x) \overset{?}{=} u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{nc\pi}{L} 1 \cdot \sin\left(\frac{n\pi x}{L}\right)$$

FOURIER SINE SERIES.

$$\frac{nc\pi}{\cancel{L}} \cdot B_n = \frac{2}{\cancel{L}} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\therefore \quad B_n = \frac{2}{nc\pi} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

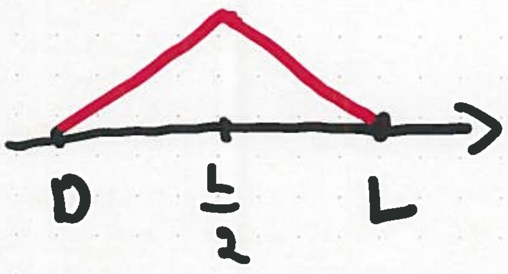
Frequency $\dfrac{\frac{cn\pi}{L}}{2\pi} = \dfrac{cn}{2L} = \lambda_n$ (n^{th} term)

cycles per unit length.

#

Ex

$$f(x) = \begin{cases} \frac{2k}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

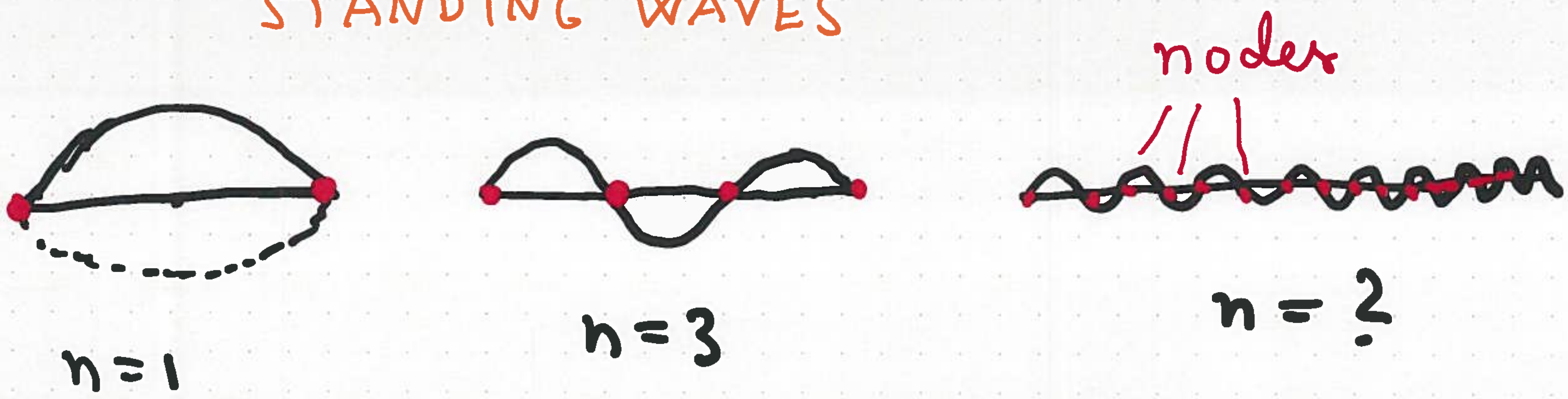


$$\begin{cases} \frac{\partial u(x,0)}{\partial t} = 0 \\ u(x,0) = f(x) \end{cases}$$

SOLUTION:

$$u(x,t) = \frac{8k}{\pi^2}\Big\{ \sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi c}{L}t\right) - \frac{1}{3^2}\sin\left(\frac{3\pi}{L}x\right)\cos\left(\frac{3\pi c}{L}t\right) + \frac{1}{5^2}\sin\left(\frac{5\pi}{L}x\right)\cos\left(\frac{5\pi c}{L}t\right) - \cdots \Big]$$

STANDING WAVES



Frequencies for a string of

$$\lambda_n = \frac{cn}{2L}$$

length L = λ_n

length $\frac{L}{2}$ = $2\lambda_n$

length $2L$ = $\frac{\lambda_n}{2}$