Ex.:
$$\begin{cases} y'' + 2y' + y = e^{-t} \\ y(0) = -1, \quad y'(0) = +1 \end{cases}$$

TAKE $\mathcal{L}\{\text{everything}\}$!

- $$\underbrace{s^2 Y(s) + s - 1}_{= \mathcal{L}\{y''\}} + 2\underbrace{\left(sY(s) + 1\right)}_{= \mathcal{L}\{y'\}} + Y(s) = \frac{1}{s+1}.$$

$$\underbrace{(s^2 + 2s + 1)}_{(s+1)^2} Y(s) = -s - 1 + \frac{1}{s+1}$$

Find $Y(s)$!

- $Y(s) = \frac{1}{(s+1)^3} - \frac{1}{s+1}$

Find $y(t)$!

- $y(t) = \frac{1}{2} t^2 e^{-t} - e^{-t}$

$$\mathcal{L}\{t^2\} = 2/s^3$$

[s-shift] $\mathcal{L}\{e^{-t} t^2\} = \frac{2}{(s+1)^3}$

<u>Answer:</u> $y(t) = \frac{1}{2} t^2 e^{-t} - e^{-t}$.

TRANSFORM of $\int_0^t f(\tau)\,d\tau$

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)}$$

$$q(t) = \int_0^t i(\tau)\,d\tau \qquad |f(t)| \leq Me^{+\beta t}$$

$$Q(s) = \mathcal{L}\{q(t)\} = \frac{1}{s}I(s)$$

Proof: $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} \overset{\text{DEF.}}{=} \int_0^\infty e^{-st}\left(\int_0^t f(\tau)\,d\tau\right)dt$

SWITCH order of integration

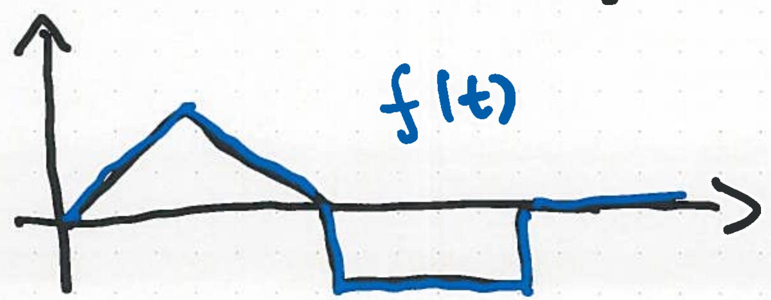
$$= \int_0^\infty\left(\int_{t=\tau}^\infty e^{-st} f(\tau)\,dt\right)d\tau$$

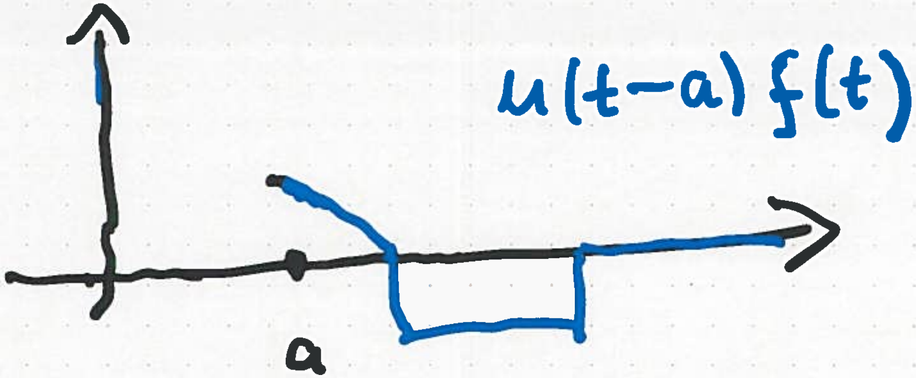
[<u>Recall</u> $\iint_{??} g(x,y)\,dx\,dy = \iint_{??} g(x,y)\,dy\,dx$]

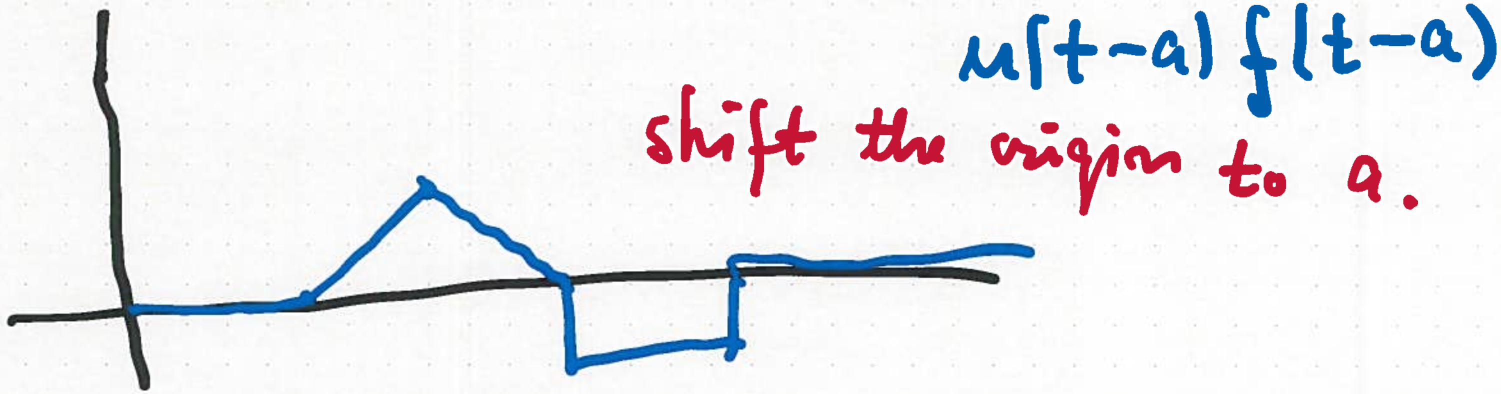
$$= \int_0^\infty f(\tau) \underbrace{\left(\left. \frac{e^{-st}}{(-s)} \right|_\tau^\infty \right)}_{e^{-s\tau}/s} d\tau = \frac{1}{s} \underbrace{\int_0^\infty e^{-s\tau} f(\tau)\, d\tau}_{F(s)} \quad \blacksquare$$

Heaviside's Function / Unit Step Function

<u>DEF.</u>: $u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$







shift the origin to a.

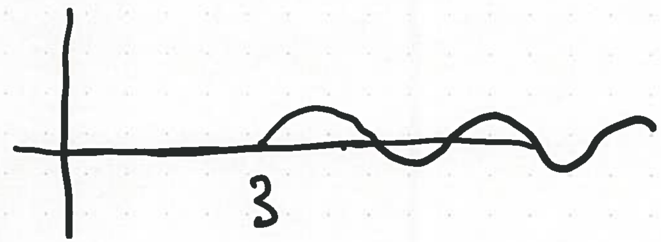
t-shifting

$$\boxed{\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)}$$

Ex.: $e^{-3s}\frac{4}{s^2+4} \longleftrightarrow 2\sin[2(t-3)]\cdot u(t-3)$

$a = 3$

$2\mathcal{L}\{\sin(2t)\} = \frac{4}{s^2+4}$



Proof: of t-shift $\mathcal{L}\{u(t-a)f(t-a)\}$

$$= \int_0^\infty u(t-a)f(t-a)e^{-st}\,dt$$

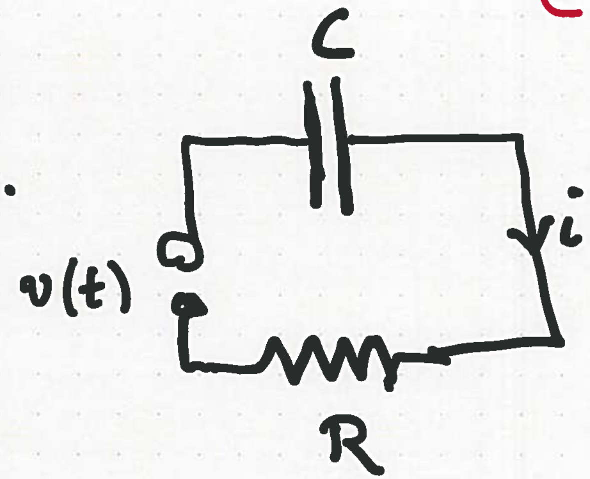
$$= \int_a^\infty 1\, f(t-a)e^{-st}\,dt$$

$t - a = \tau$

$dt = d\tau$

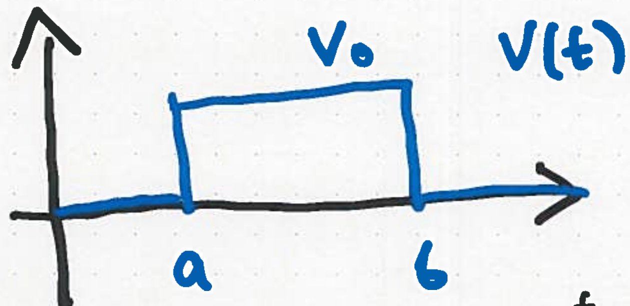
$$= \int_0^\infty f(\tau)\, \underbrace{e^{-s(\tau+a)}}_{e^{-s\tau} e^{-as}}\, d\tau = e^{-as} F(s) \quad \square$$

Ex.



$q(t) = \int_0^t i(\tau)\,d\tau$ [Charge]

[$i(t)$ = current]



$v(t) = V_0 \left[u(t-a) - u(t-b) \right]$

V_0 = constant

- $R i(t) + \frac{1}{C} \int_0^t i(\tau)\,d\tau = v(t)$ $\qquad$ $i(t) = ?$

$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$ [Verify it!]

$$R\,I(s) + \frac{1}{C}\underbrace{\mathcal{L}\left\{\int_0^t i(\tau)\,d\tau\right\}}_{\frac{1}{s}I(s)} = \frac{V_0}{s}\left[e^{-as} - e^{-bs}\right]$$

$$\left(R + \frac{1}{Cs}\right) I(s) = \frac{V_0}{s}\left[e^{-as} - e^{-bs}\right]$$

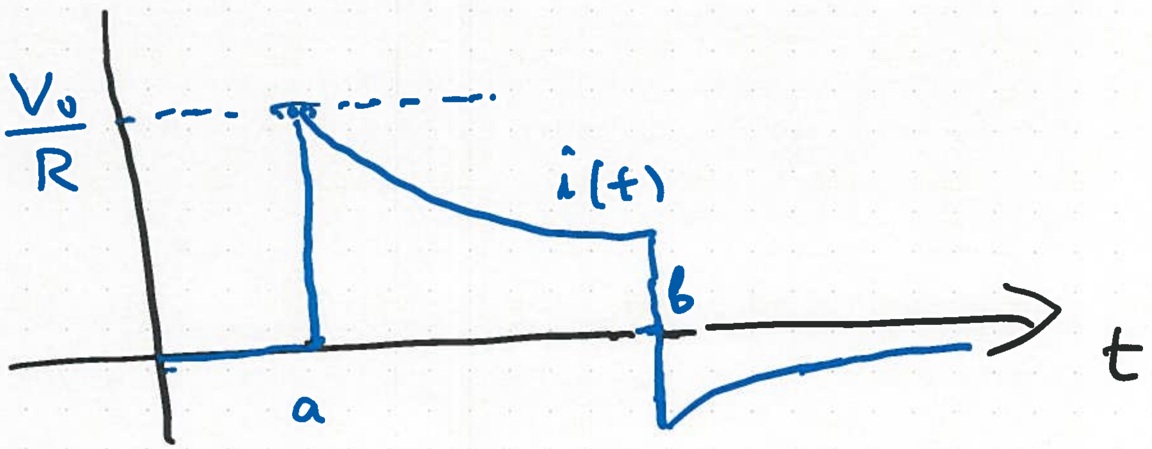
$$I(s) = \frac{V_0}{s\left(R + \frac{1}{Cs}\right)}\left[e^{-as} - e^{-bs}\right]$$

$$= \frac{V_0}{R\left(s + \frac{1}{RC}\right)}\left[e^{-as} - e^{-bs}\right]$$

Suggests t-shift.

$$\mathcal{L}\left\{e^{-t/RC}\right\} = \frac{1}{s + \frac{1}{RC}}$$

$$i(t) = \frac{V_0}{R}\left[u(t-a)\,e^{-\frac{t-a}{RC}} - u(t-b)\,e^{-\frac{t-b}{RC}}\right]$$



PARTIAL FRACTION DECOMPOSITION

Recall $\displaystyle\int \frac{x+1}{(x^2+1)(x-1)x}\,dx = ?$

$$\frac{x+1}{(x^2+1)(x-1)x} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}.$$

<u>Ex.:</u> $\begin{cases} y'' + 4y' + 13y = 20e^{-t} \\ y(0) = 1, \quad y'(0) = 3 \end{cases}$

- $s^2 Y(s) - s \cdot 1 - 3 + 4[sY(s) - 1] + 13Y(s)$

$$= \frac{20}{s+1}$$

- $Y(s) = \dfrac{s^2 + 8s + 27}{(s+1)(s^2 + 4s + 13)}$

$$= \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4s + 13}$$

$$= \frac{2}{s+1} + \frac{-s + 1}{(s+2)^2 + 9}$$

$$= \frac{2}{s+1} - \frac{s+2}{(s+2)^2+3^2} + \frac{3}{(s+2)^2+3^2}$$

$$y(t) = 2e^{-t} - e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$$

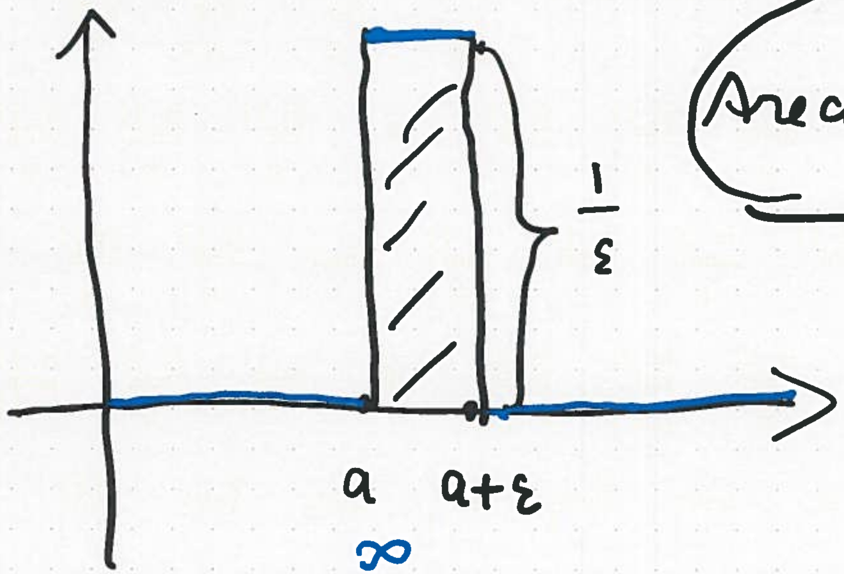
$$\mathcal{L}\{\cos(3t)\} = \frac{s}{s^2+3^2} \qquad s\text{-shift.}$$

DIRAC's DELTA $\delta(t-a)$

$$\text{"}\int_0^\infty \delta(t-a)\, f(t)\, dt\text{"} = f(a)$$

"DEFINITION":
$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ \infty, & t = a \end{cases}$$

VERY INFINITE!



Area = 1

$\delta_\varepsilon(t-a)$ = PICTURE

SUDDEN IMPULSE !
centered at $t = a$

$$\lim_{\varepsilon \to 0+} \int_0^\infty \delta_\varepsilon(t-a) f(t)\, dt = f(a)$$

$$\boxed{\delta(t-a) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t-a)}$$