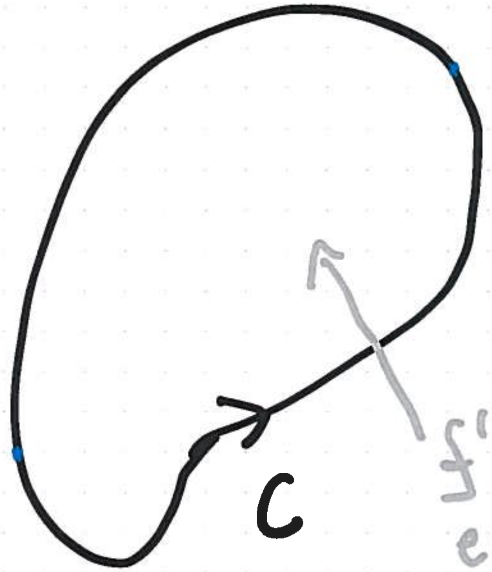


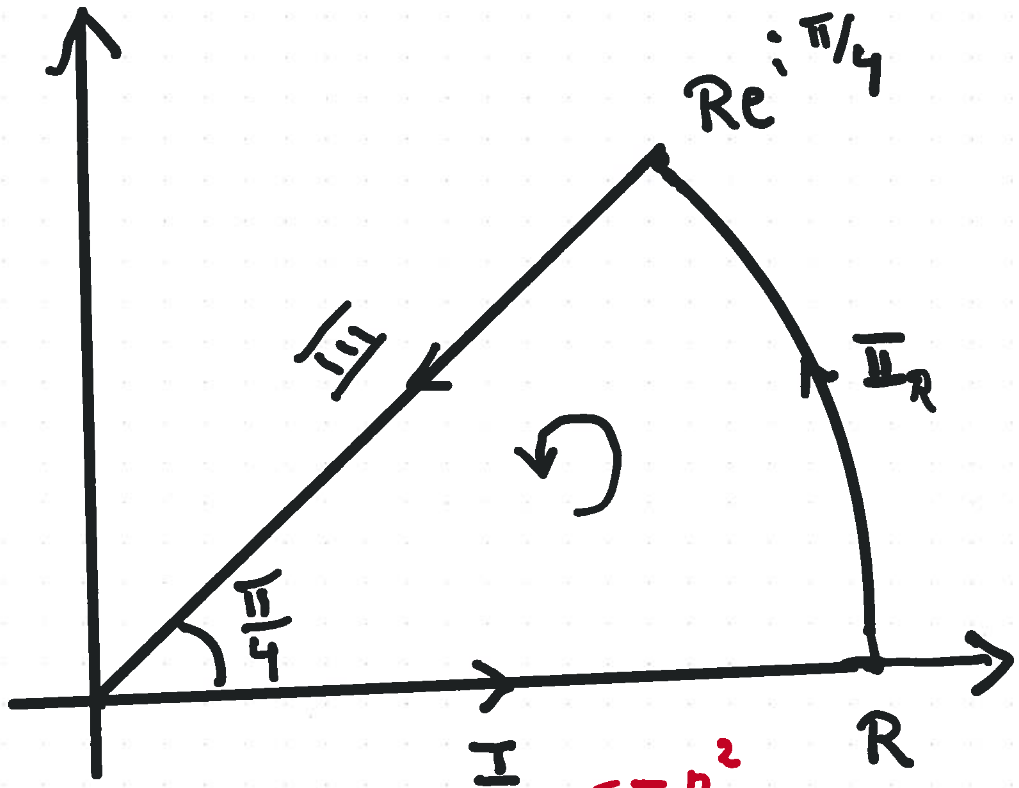
$f'(z)$ exists at each point inside the loop.

$$\boxed{\oint_C f(z)\,dz = 0}$$

Ex. $\oint_C e^{z^2}\cos(z)\,dz = 0.$

Ex. FRESNEL INTEGRALS $\int_0^\infty \cos(x^2)\,dx$, $\int_0^\infty \sin(x^2)\,dx$.

$$\underline{\int_0 e^{ix^2}dx} = \int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx$$



$$0 = \oint e^{iz^2}\,dz$$

$$= \int_0^R e^{ix^2}\,dx$$

$z = x$

$dz = dx$

$$+ \int_0^{\pi/4} e^{i(Re^{i\theta})^2} \underbrace{iRe^{i\theta}\,d\theta}_{dz}$$

$z = Re^{i\theta}$

$dz = iRe^{i\theta}\,d\theta$

$$+ \int_R^0 e^{\underbrace{i r^2 e^{i\pi/2}}_{=-r^2}} e^{i\frac{\pi}{4}}\,dr$$

$z = re^{i\pi/4}$

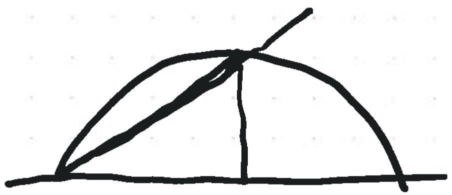
$dz = e^{i\pi/4}\,dr$

The integral along the circular arc approaches 0 as $R \to \infty$. To see this, note

$$e^{iR^2(\cos(2\theta)+i\sin(2\theta))} = e^{-iR^2\cos(2\theta)}\, e^{-R^2\sin(2\theta)}$$

$$\left| e^{iR^2(\cos(2\theta)+i\sin(2\theta))} \right| = e^{-R^2\sin(2\theta)} \overset{*)}{\leq} e^{-R^2\frac{2}{\pi}\cdot 2\theta}$$

*) $0 < 2\theta < \frac{\pi}{2}$



$\sin(\alpha) > \frac{2}{\pi}|\alpha|$ when $0 < \alpha < \frac{\pi}{2}$

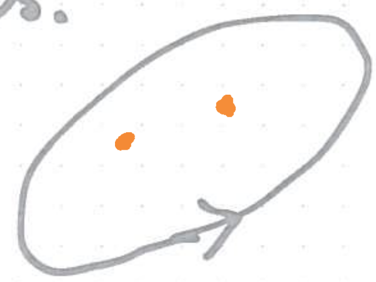
$$\left| \int_{\Pi_R} \cdots \right| \leq \int_0^{\pi/4} e^{-R^2\frac{2}{\pi}(2\theta)} R\, d\theta \xrightarrow[R\to\infty]{} 0.$$

[Can be evaluated]

$$\Rightarrow \int_0^\infty e^{ix^2}dx = e^{i\frac{\pi}{4}} \int_0^\infty e^{-r^2}dr = \frac{i+1}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2}$$

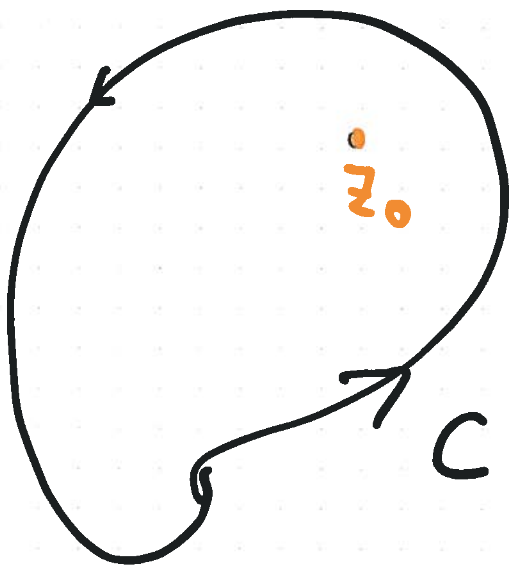
Answer $\displaystyle\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$ #

Later: Calculus of Residues.



CAUCHY'S FORMULA

$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz}$$



IF $f(z)$ IS ANALYTIC INSIDE THE LOOP, THEN THE FORMULA HOLDS TRUE, PROVIDED THAT z_0 IS strictly INSIDE.

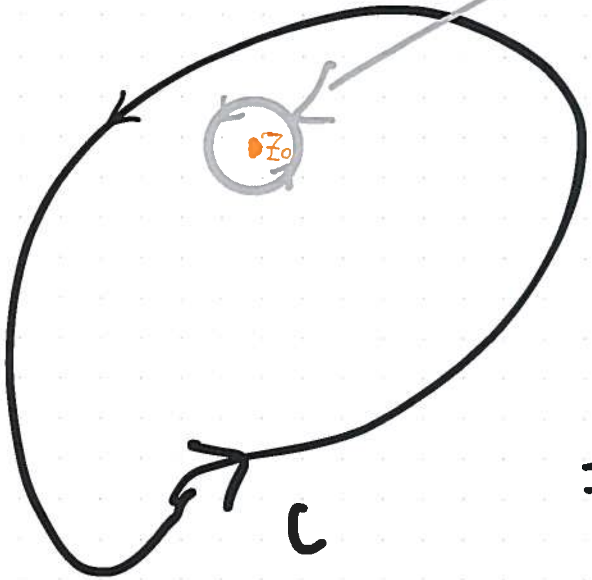
Ex

$$1 = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^z}{z} \, dz$$

$$e^0 = \frac{1}{2\pi i} \oint_{|z|=1} \frac{e^z}{z-0} \, dz$$

PROOF.

$|z-z_0| = \rho$, ρ = small.



Deformation of paths

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_{|z-z_0|=\rho} \frac{f(z)}{z-z_0}\,dz$$

$$= \underbrace{\oint_{|z-z_0|=\rho} \frac{f(z_0)\,dz}{z-z_0}}_{2\pi i f(z_0)} + \underbrace{\oint_{|z-z_0|=\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz}_{\text{This is, in fact, } = 0.}$$

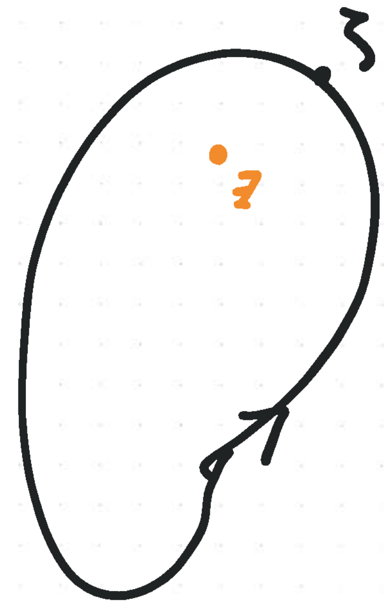
Indeed,

$$\left| \oint \frac{f(z_0)-f(z)}{z-z_0}\,dz \right| \leq \oint \frac{|f(z_0)-f(z)|}{\underbrace{|z-z_0|}_{\rho}}\,|dz|$$

$$\leq \max_{\substack{|z-z_0|=\rho \\ z}} |f(z_0)-f(z)| \cdot \frac{1}{\cancel{\rho}} \cdot 2\pi \cancel{\rho}$$

$$\longrightarrow 0 \text{ as } \rho \to 0+. \qquad \square$$

HIGHER DERIVATIVES



$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$f'(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

$$\frac{d}{dz} (\zeta - z)^{-1} = (\zeta - z)^{-2}$$

$$f''(z) = \frac{2}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^3} d\zeta$$

$$f'''(z) = \frac{2\cdot 3}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^4}\, d\zeta$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

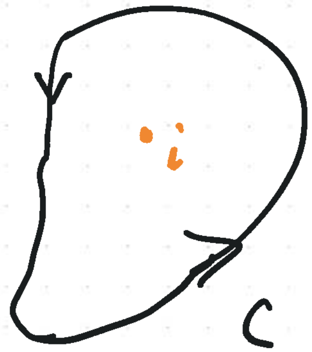
$$\vdots$$

! Hence an analytic function has infinitely many derivatives! **!**

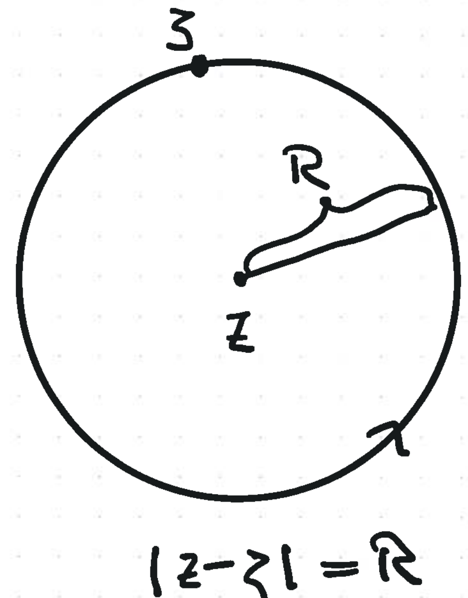
<u>Ex.:</u>

$$\oint_C \frac{\sin(z)}{(z-i)^3}\, dz = \frac{2\pi i}{2}\left[\frac{d^2}{dz^2}\sin(z)\right]_i$$

$$= \pi i\left[-\sin(i)\right] = \pi \sinh(1).$$



THM (LIOUVILLE) Suppose that $f(z)$ is analytic and bounded in the entire complex plane. Then $f(z)$ reduces to a constant.



<u>Proof</u> Assume: $|f(z)| \le M$ for all z.

$$f'(z) = \frac{1}{2\pi i} \oint_{|\zeta - z| = R} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

ML-ineq.

$$|f'(z)| \le \frac{1}{2\pi} \cdot M \frac{1}{R^2} \cdot 2\pi R$$

$$= \frac{M}{R} \longrightarrow 0 \quad \text{as } R \to \infty$$

$$\Longrightarrow |f'(z)| = 0 \Longrightarrow f'(z) = 0$$

Thus $f'(z) = 0$ at every point. $\Longrightarrow f(z) = \text{Const.}$ $\square$

[§17 Kreyszig] CONFORMAL MAPPING

$$f = u + iv$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j}$$

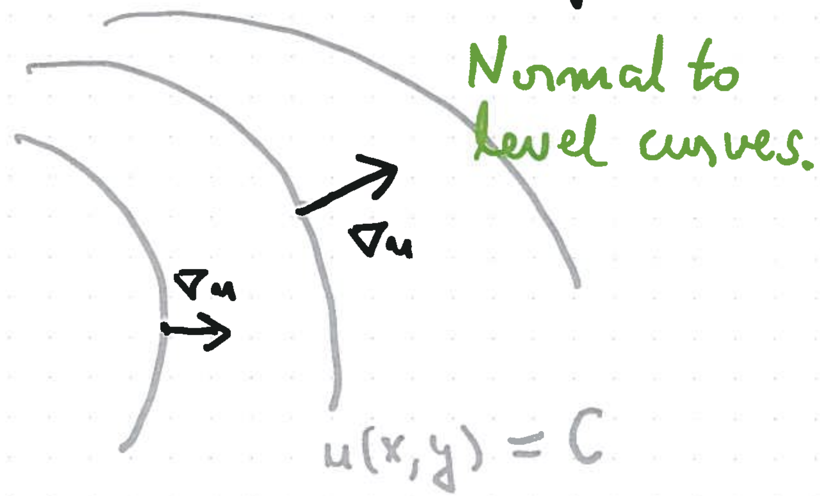
$$\nabla v = \frac{\partial v}{\partial x}\hat{i} + \frac{\partial v}{\partial y}\hat{j}$$

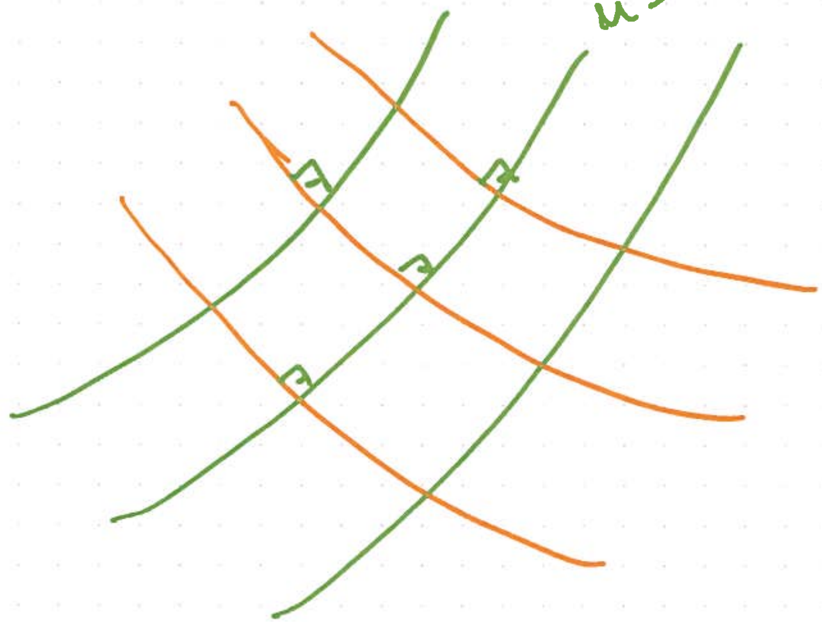
$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y$$

$$= 0 \quad \text{(Cauchy–R.)}$$

$$\boxed{\nabla u \perp \nabla v}$$

Orthogonality

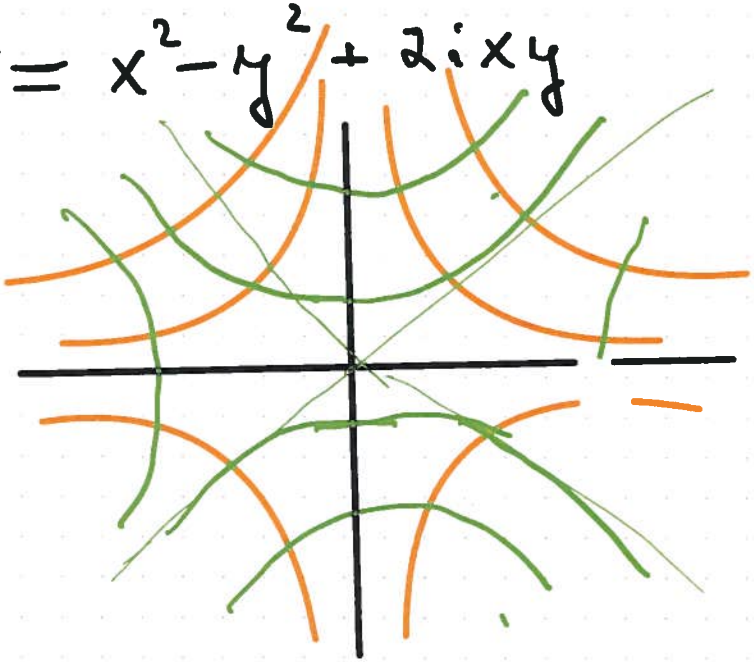




Level curves intersect orthogonally.

Ex $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + 2ixy$

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

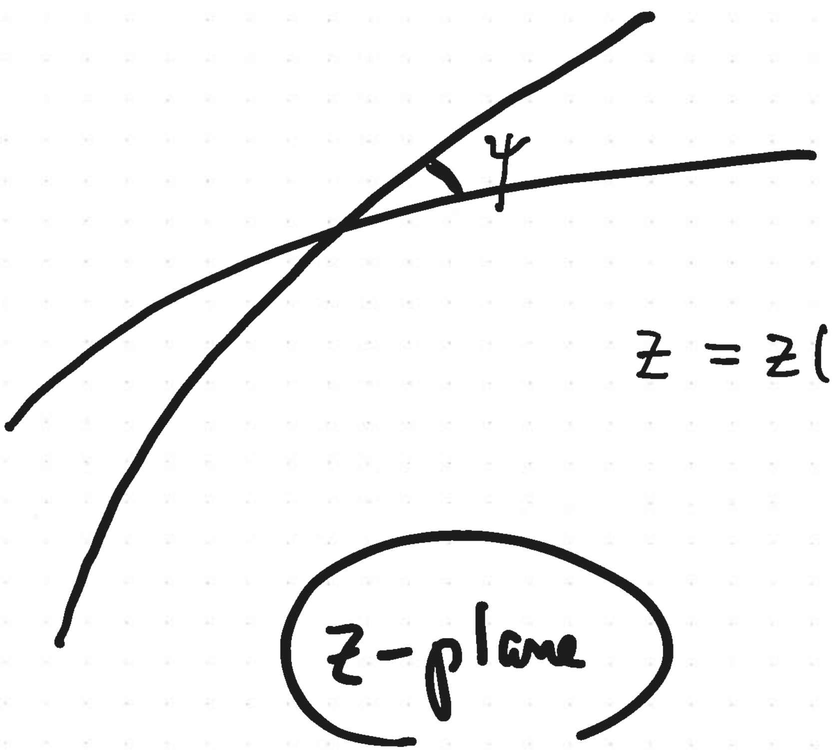


DEF. $f(z)$ is conformal $\iff f'(z) \neq 0$.

JACOBIAN

$$\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2 > 0$$

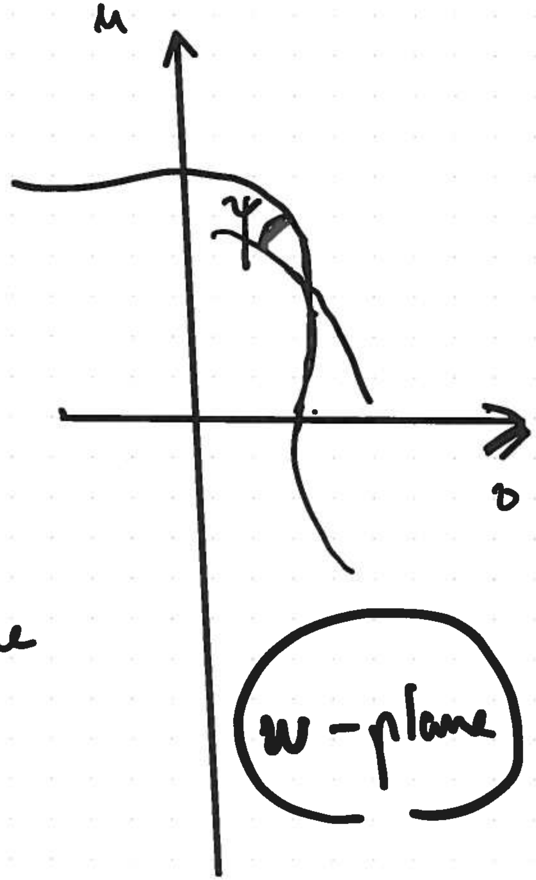
ANGLES PRESERVED

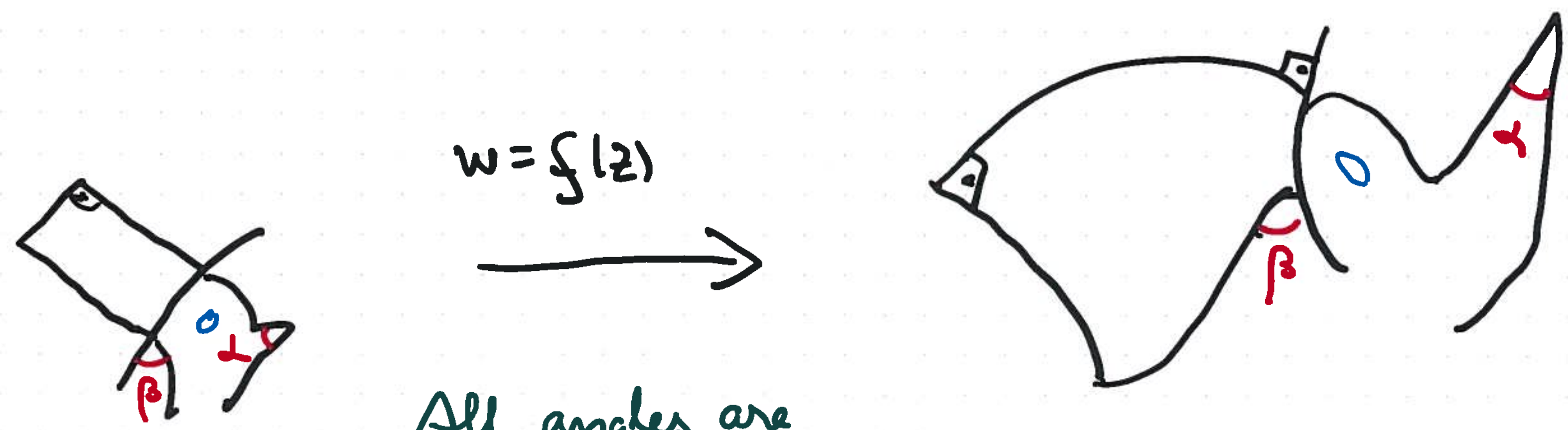


$z = z(t)$ Image curve

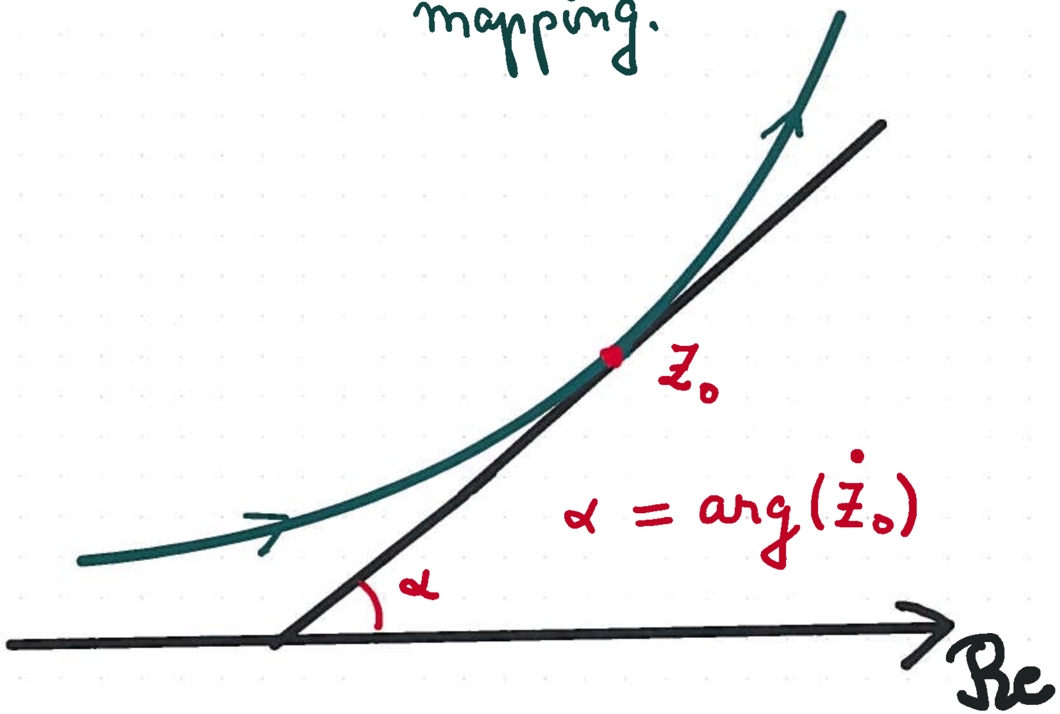
$w = u + iv$

$= f(z(t))$





All angles are preserved under a conformal mapping.



$$z = z(t)$$
$$z_0 = z(t_0)$$
$$\dot{z} = \dot{z}(t)$$
$$\dot{z}_0 = \dot{z}(t_0)$$