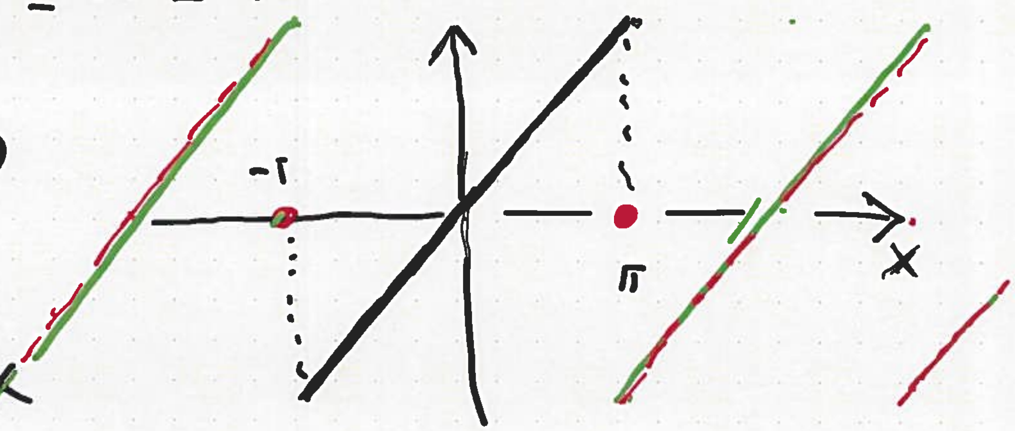
Ex. $f(x) = x \,, \quad -\pi \le x \le \pi$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = \cdots = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

Answer:

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(nx)}{n} \underset{\text{THM below.}}{=} \begin{cases} x, & |x| < \pi \\ 0, & x = \pm\pi \end{cases}$$

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DEF. The function $f(x)$ is piecewise differentiable in $[a,b]$ if it differentiable, except possibly at a finite number of exceptional points. At each exceptional point c it is required that

1°) the finite limits

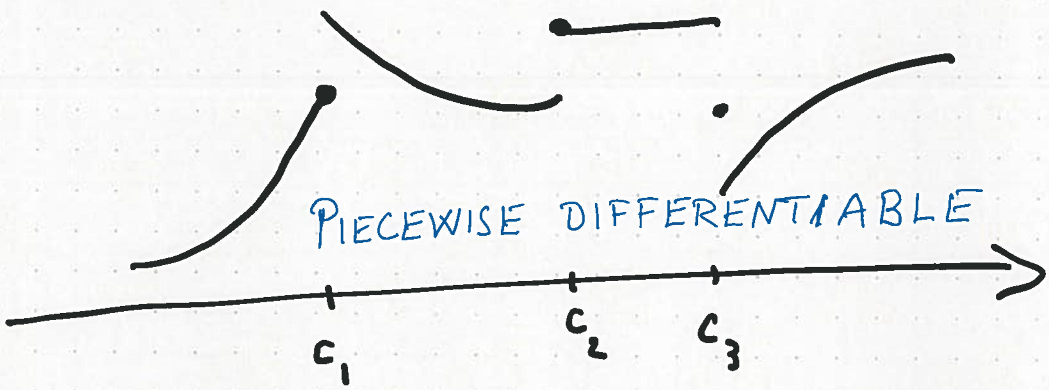
$$\begin{cases} f(c+0) = \lim\limits_{x\to c+} f(x) & \text{[right]} \\ f(c-0) = \lim\limits_{x\to c-} f(x) & \text{[left]} \end{cases}$$

exist

2°) the one-sided derivatives

$$\begin{cases} \lim\limits_{h\to 0+} \dfrac{f(c+h)-f(c+0)}{h} = f_+'(c) \\ \lim\limits_{h\to 0-} \dfrac{f(c+h)-f(c-0)}{h} \end{cases}$$

exist.



THEOREM Suppose that the function $f(x)$ is piecewise differentiable in $[-\pi,\pi]$. Then the Fourier series of $f(x)$,

$$S(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

converges at every point. The sum is:

- $S(x) = f(x)$, if $|x| < \pi$ and $f(x)$ is continuous at the point x

- $S(x) = \dfrac{f(x-0) + f(x+0)}{2}$, if $|x| < \pi$ and $f(x)$ is discontin. at x.

- $S(\pm\pi) = \dfrac{f(\pi - 0) + f(-\pi + 0)}{2}$ [Endpoints!]

PERIOD $\omega = 2L$ (L replaces π)

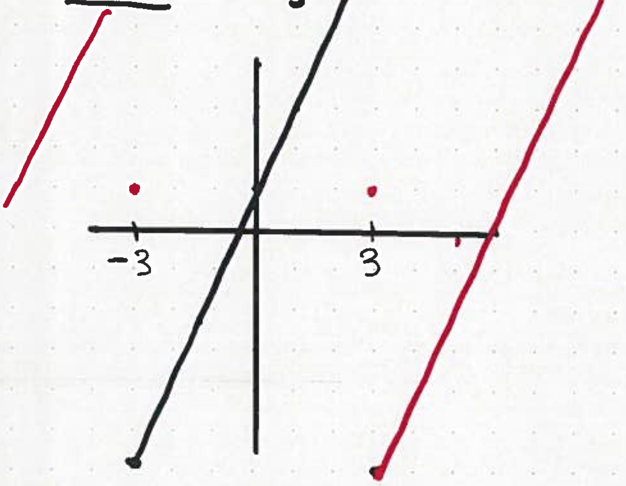
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\begin{cases} a_0 = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x)\,dx & \text{[average]} \\[2ex] a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos\left(\dfrac{n\pi x}{L}\right) dx \end{cases}$$

$$\left[\; b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \right. \qquad *$$

Ex. $f(x) = 2x+1 \;, \quad -3 \le x \le 3 \;; \quad L = 3$

Period $= 6$



$$a_0 = \frac{1}{6}\int_{-3}^{3}(2x+1)\,dx = 1$$

$$a_n = \frac{1}{3}\int_{-3}^{3}(2x+1)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= 0, \quad n \ge 1.$$

$$b_n = \frac{1}{3}\int_{-3}^{3}(2x+1)\sin\left(\frac{n\pi x}{3}\right)dx = \cdots = -\frac{12}{n\pi}(-1)^n$$

$$2x+1 = 1 + \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{3}\right), \quad |x| < 3$$

At $x = \pm 3$, the sum is $\frac{7+(-5)}{2} = 1$. #

EVEN / ODD FUNCTIONS

- $f(x) \equiv -f(x)$ $\quad f(x)$ is EVEN $\qquad \cosh(x)$
- $f(x) \equiv -f(-x)$ $\quad f(x)$ is ODD $\qquad \sinh(x)$

EVEN $\;1, x^2, x^4, x^6, \ldots;\; \cos(x), \cos(2x), \cos(3x), \ldots$

ODD $x, x^3, x^5, x^7, \ldots;$ $\sin(x), \sin(2x), \sin(3x), \ldots$

$$\left\{\begin{array}{lcl} \text{EVEN} \cdot \text{EVEN} & = & \text{EVEN} \\ \text{ODD} \cdot \text{ODD} & = & \text{EVEN} \\ \text{ODD} \cdot \text{EVEN} & = & \text{ODD} \\ \text{EVEN} \cdot \text{ODD} & = & \text{ODD} \end{array}\right.$$

multiplication

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

- If $f(x)$ is EVEN, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

and $$a_0 = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{\pi} f(x)\,dx$$ ONLY COSINES!

- If $f(x)$ is ODD, then

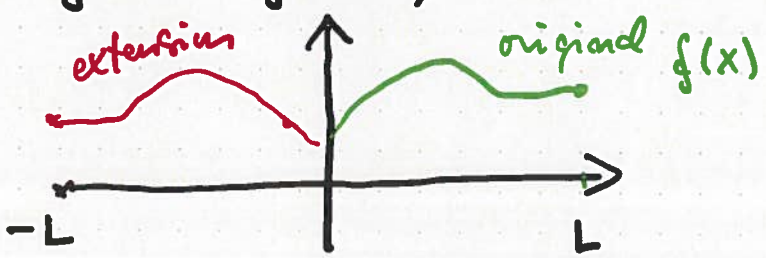
$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

ONLY SINES

HALF-RANGE EXPANSIONS

Given $f(x)$, when $0 \le x \le L$

The COSINE series of $f(x)$ is the Fourier series of the EVEN EXTENSION of $f(x)$, i.e. $f(x) = f(-x)$, when $-L \le x \le 0$.

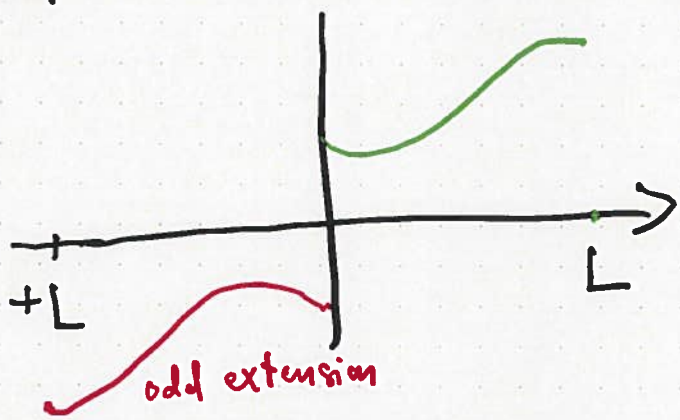


$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_n = \frac{2}{\pi}\int_0^L f(x)\cos(nx)\,dx$$

The SINE series of $f(x)$ is the Fourier series of the ODD extension to $-L \le x \le L$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi}\int_0^L f(x)\sin(nx)\,dx$$



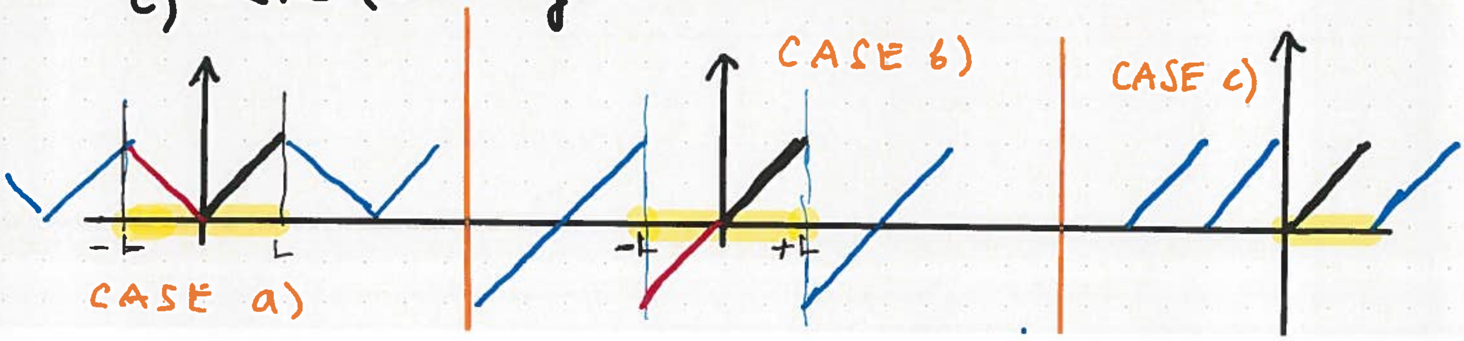
[period = 2L]

<u>Ex</u> Let $f(x) = x$, when $0 \le x \le L$. Find

a) The cosine series [period = 2L]

b) The sine series [period = 2L]

c) The (ordinary) Fourier series. [Period = L]



a) $a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$

$b_n = 0$

b) $b_n = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$

$a_n = 0$

c) See below!

CASE c)

period $= L$, (not $2L$)

$$a_n = \frac{1}{\left(\frac{L}{2}\right)}\int_0^L x\cos\left(\frac{n\pi x}{\frac{L}{2}}\right)dx$$

$$b_n = \frac{1}{\left(\frac{L}{2}\right)}\int_0^L x\sin\left(\frac{n\pi x}{\frac{L}{2}}\right)dx$$

In the case $L=\pi$ a calculation yields:

a) $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \cdots\right).$

b) $2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \cdots\right).$

c) $\frac{\pi}{2} - \frac{\sin(2x)}{1} - \frac{\sin(4x)}{2} - \frac{\sin(6x)}{3} - \cdots.$

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