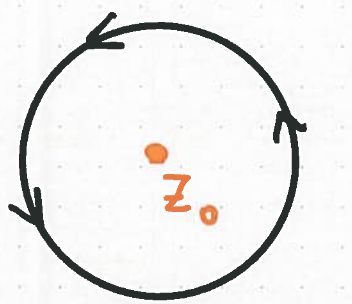
Ex

$$\oint_{|z-z_0|=R} \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{Rie^{i\theta}\,d\theta}{R^n e^{in\theta}} = \dagger)$$



$|z-z_0|=R$

$$z = z_0 + Re^{i\theta} \quad (0 \le \theta \le 2\pi)$$

$$dz = Rie^{i\theta}\,d\theta \qquad n = 0, \pm 1, \pm 2, \dots$$

$$\dagger) = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta}\,d\theta$$

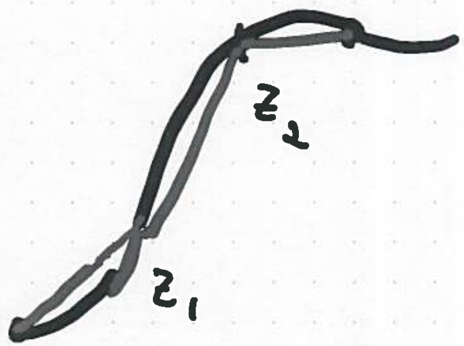
$$= \begin{cases} \dfrac{i}{R^{n-1}} \Big/_0^{2\pi} \dfrac{e^{i(1-n)\theta}}{i(1-n)} = 0, & n \neq 1 \\ i \displaystyle\int_0^{2\pi} 1\,d\theta = \underline{2\pi i}, & \underline{n = 1} \end{cases}$$

ANSWER $$\oint \frac{dz}{(z-z_0)^n} = \begin{cases} \underline{2\pi i}, & n = 1 \\ 0, & n = 0, -1, \pm 2, \pm 3, \dots \end{cases}$$

ML - INEQUALITY.

$$\boxed{\left|\int_C f(z)\,dz\right| \leq M \cdot L}$$

where $|f(z)| \leq \underline{M}$ on C and $L = \int_C |dz|$ = length of the curve C.



Proof:

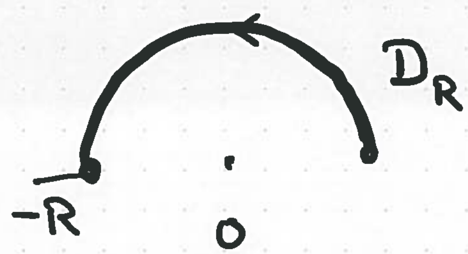
$$\left|\sum_k f(\zeta_k)(z_k - z_{k-1})\right| \leq$$

$$\sum |f(\zeta_k)(z_k - z_{k-1})| = \sum \underbrace{|f(\zeta_k)|}_{\leq M} |z_k - z_{k-1}|$$

$$\leq M \underbrace{\sum |z_k - z_{k-1}|}_{\text{length of polygon}} \leq M \cdot L \quad \square$$

Ex. $I_R = \displaystyle\int_{D_R} \frac{e^{i3z}}{z^2+9}\,dz$

Half-circle in the upper half-plane.



Half-Circle

$z = Re^{i\theta}, \quad 0 \le \theta \le \pi.$

$dz = iRe^{i\theta}\,d\theta$

CLAIM $\displaystyle\lim_{R\to\infty} I_R = 0$

$$e^{3iz} = e^{3i(\cos\theta + i\sin\theta)R}$$
$$= e^{3iR\cos\theta}\, e^{-iR\sin\theta}$$

$$|e^{3iz}| = e^{-3\sin(\theta)\cdot R} \le 1 \quad \text{when } 0 \le \theta \le \pi$$

$$|z^2+9| \ge |z^2| - 9 = |z|^2 - 9 \ge R^2 - 9$$

$$\left|\frac{e^{3iz}}{z^2+9}\right| \le \frac{1}{R^2-9}, \quad R > 3.$$

$$\left|\int_{\Gamma} \frac{e^{3iz}\,dz}{z^2+9}\right| \overset{ML}{\leq} \frac{1}{R^2-9}\cdot \pi R \longrightarrow 0 \quad \text{as } R \to \infty \qquad \#$$

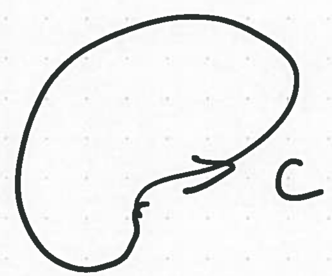
TRIANGLE INEQUALITY.

<u>COMMENT</u> $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$

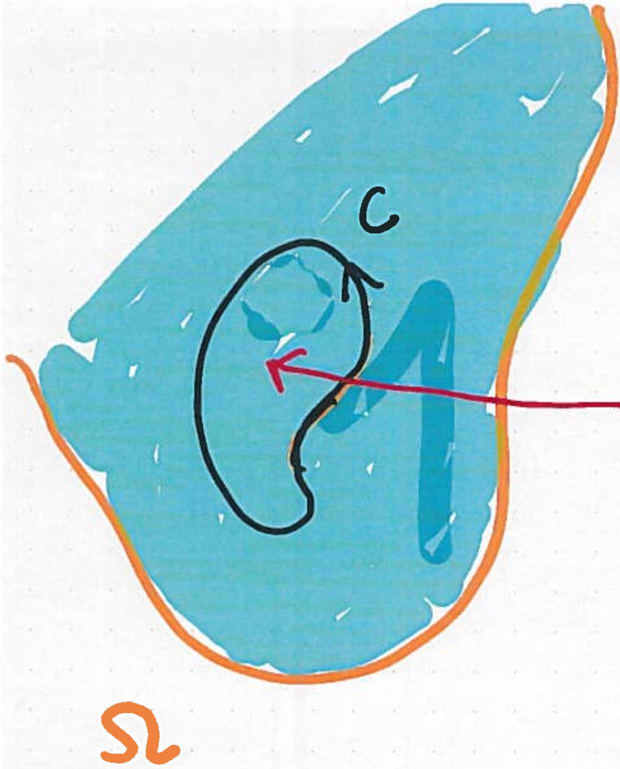
CAUCHY'S INTEGRAL THM:

If $f(z)$ is analytic in a <u>SIMPLY</u> connected domain Ω, then

$$\oint_C f(z)\,dz = 0$$



along any (simple) closed curve ("contour", loop, circuit) C inside Ω.



$$\oint_C f(z)\,dz = 0$$

$f(z)$ is analytic at EVERY point inside the domain bounded by the loop C.

ILLUSTRATION $\quad f(z) = \dfrac{1}{z^2+1}$ analytic when $z \neq \pm i$

- $\oint_{C_1} f(z)\,dz = 0$ by CAUCHY'S THM.
- $\oint_{C_2} f(z)\,dz = \pi$ (NOTE $z = i$ inside the loop)

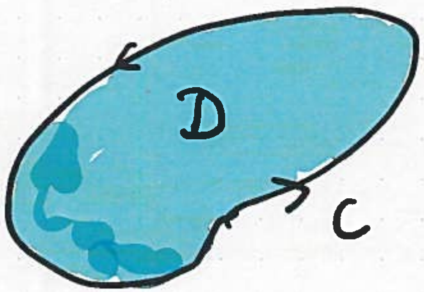
- $\oint_{C_3} f(z)\,dz = -\pi$, • $\oint_{C_4} f(z)\,dz = 0$

<u>Ex:</u> $\oint_C z^2\,dz = 0$, $\oint_C e^{z^2}\cos(z)\,dz = 0$

PROOF under the assumption that $f'(z)$ is continuous. Recall

$$\oint_C P\,dx + Q\,dy = \iint_D \underbrace{\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]}_{\text{continuous!}} dx\,dy$$

GREEN



$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C v\,dx + u\,dy$$

$P = u$
$Q = -v$

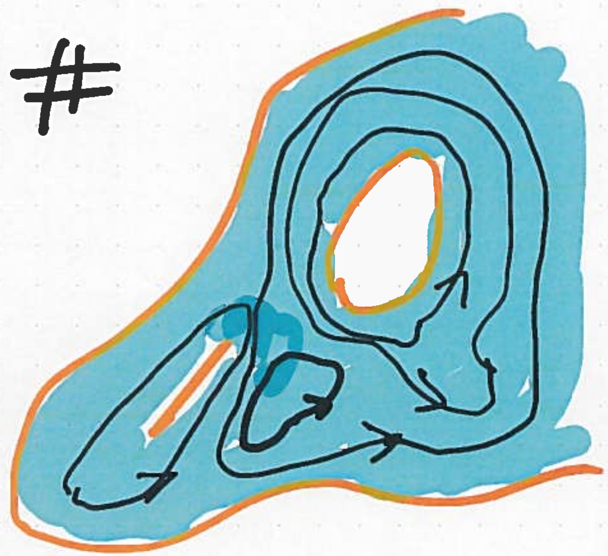
$$\overset{\text{GREEN}}{=} \iint_\Omega \underbrace{\left[\frac{\partial}{\partial x}(-v) - \frac{\partial u}{\partial y}\right]}_{=0} dx\,dy + i\iint_\Omega \underbrace{\cdots}_{=0}$$

Cauchy-Riemann

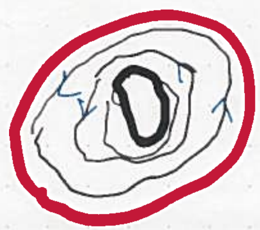
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$= 0$ Cauchy-Riemann.

#



DEFORMATION OF PATHS

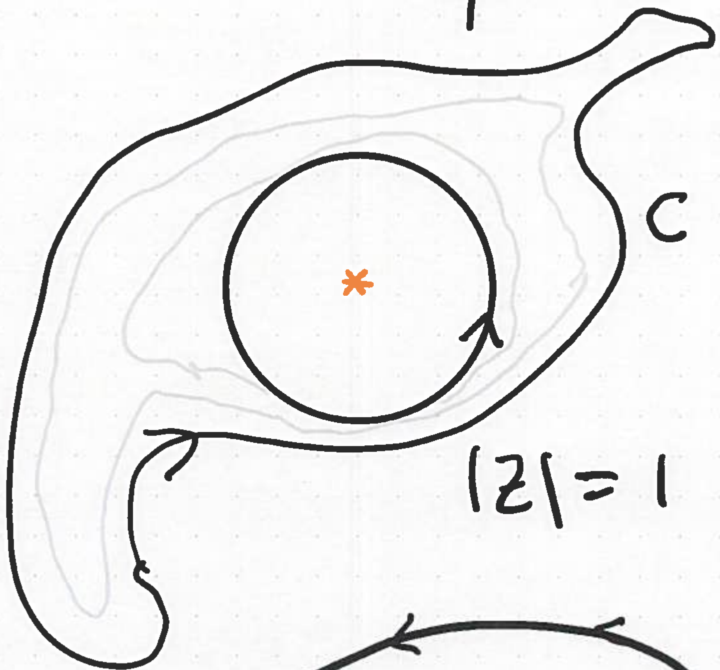


$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz \qquad C_1 \underset{\text{deformation}}{\overset{\text{Cont}}{\longleftrightarrow}} C_2$$

<u>Ex:</u>

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i$$

$$\Longrightarrow \int_{C} \frac{dz}{z} = 2\pi i$$

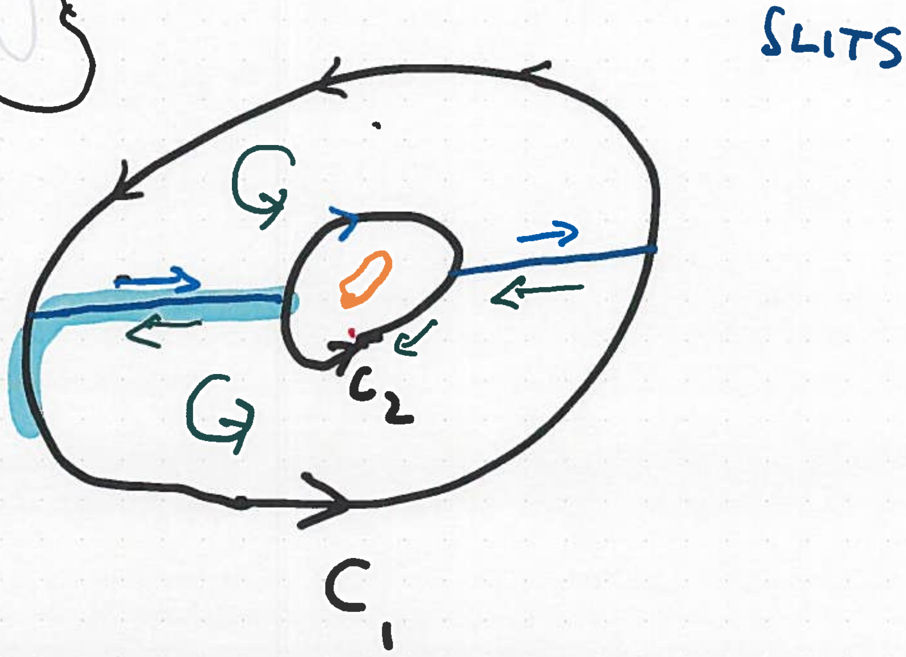


We deform C to become a circle!

EXPLANATION

<u>**Claim:**</u>

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



$$\int f(z)\,dz = 0 \quad (\text{CAUCHY})$$

$$\int f(z)\,dz = 0 \quad (\text{CAUCHY})$$

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

clockwise

Change direction!

$+ -$

$$\Longrightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

counterclockwise

<u>Ex</u> $I = \oint_C \frac{dz}{z^2+1}$

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \left(\frac{1}{z-i} - \frac{1}{z+i}\right)\frac{1}{2i}$$

C is deformed to a circle.

$$I = \frac{1}{2i}\left[\int_C \frac{dz}{z-i} - \underbrace{\oint_C \frac{dz}{z+i}}_{=0 \text{ (Cauchy)}}\right] = \frac{1}{2i}\oint_C \frac{dz}{z-i}$$

$$\overset{\text{deformation}}{=} \frac{1}{2i}\oint_{|z-i|=1} \frac{dz}{z-i} = \frac{1}{2i}\cdot 2\pi i = \pi.$$

COMMENT. $f'(z)$ exists in Ω $\xRightarrow{\text{GOURSAT 1900}}$
$f'(z)$ is continuous in Ω. $\xRightarrow{\text{CAUCHY}}$ All derivatives
$f'(z), f''(z), f'''(z), \dots$ exist $\xRightarrow{\text{CAUCHY}}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \quad \text{(Taylor)}$$