PLANCHEREL'S FORMULA

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega .$$

How to obtain the formulas as $L \longrightarrow \infty$.

Start from

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi x}{L}} , \quad c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi x}{L}} dx$$

$$\Rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} e^{i \frac{n\pi x}{L}} \boxed{\frac{1}{2L} \int_{-L}^{L} f(y) e^{-\frac{i n\pi y}{L}} dy} \quad (*)$$

Introduce the "new" function

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\omega y} dy$$

At $\omega = \frac{n\pi}{L}$, $n = 0, \pm 1, \pm 2, \dots$, we have

$$\sqrt{2\pi}\,\hat{f}\left(\frac{n\pi}{L}\right) = \int_{-\infty}^{\infty} f(y)\, e^{-\frac{in\pi}{L}y}\, dy$$

$$\approx \boxed{\int_{-L}^{L} f(y)\, e^{-\frac{in\pi}{L}y}\, dy}$$

L very large, $f(y)$ rapidly decaying as $y \to \pm\infty$.

Insert this into (*).

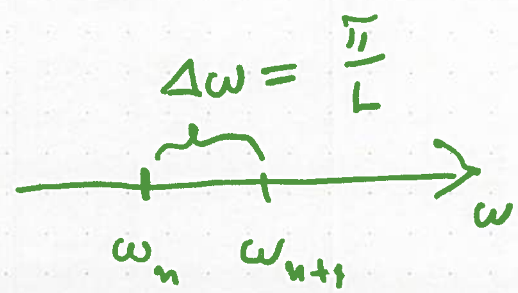
$$f(x) \approx \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{i\frac{n\pi x}{L}}\, \hat{f}\left(\frac{n\pi}{L}\right) \cdot \frac{\pi}{L}$$

RIEMANN SUM OF AN INTEGRAL.

<u>Notation:</u>

$$\omega_n = \frac{n\pi}{L}, \quad \Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$$

$$f(x) \approx \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{i\omega_n x}\, \hat{f}(\omega_n)\, \Delta\omega$$



$$\xrightarrow[L\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, d\omega$$ [Definition of integral via Riemann sums.]

The formulas become exact, when $L = \infty$. Thus

$$\left.\begin{aligned} &(1) \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \\ &(2) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} \, dx \end{aligned}\right\}$$

We deduced that (1) $\Rightarrow$ (2).

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Ex.: $\mathcal{F}\{e^{-b|x|}\} = \widehat{e^{-b|x|}}$ $(b > 0)$

$$\widehat{e^{-b|x|}} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} \underbrace{e^{bx} e^{-i\omega x}}_{e^{x(b-i\omega)}} \, dx + \int_{0}^{\infty} \underbrace{e^{-bx} e^{-i\omega x}}_{e^{-x(b+i\omega)}} \, dx \right\}$$

$$\xrightarrow[L\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$$ [Definition of integral via Riemann sums.]

The formulas become exact, when $L = \infty$. Thus

$$(1) \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

$$(2) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, dx$$

We deduced that $(1) \Rightarrow (2)$.

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$$\widehat{e^{-b|x|}} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} \underbrace{e^{bx} e^{-i\omega x}}_{e^{x(b-i\omega)}}\, dx + \int_{0}^{\infty} \underbrace{e^{-bx} e^{-i\omega x}}_{e^{-x(b+i\omega)}}\, dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-bx}\left(e^{i\omega x} + e^{-i\omega x}\right) dx$$

$$(b - i\omega)(b + i\omega) = b^2 + \omega^2$$

$$= \frac{1}{\sqrt{2\pi}} \Bigg/_{0}^{\infty} \frac{e^{x(i\omega - b)}}{i\omega - b} + \frac{e^{-x(i\omega + b)}}{-i\omega - b}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{-b + i\omega} + \frac{-1}{-b - i\omega} \right] = \sqrt{\frac{2}{\pi}} \frac{b}{\omega^2 + b^2}$$

OBS! $\left|e^{-xb \pm i\omega x}\right| = \left|e^{-xb} e^{\pm i\omega x}\right| = e^{-xb} \underbrace{\left|e^{\pm i\omega x}\right|}_{=1}$

$= e^{-xb} \xrightarrow[x \to \infty]{} 0 \quad (b > 0)$

<u>Inverse</u>

$$e^{-b|x|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{b e^{+i\omega x}}{\omega^2 + b^2} d\omega$$

RULES

LINEARITY:

$$\widehat{af(x) + bg(x)} = a\widehat{f(x)} + b\widehat{g(x)}$$

$$\mathcal{F}\{af(x) + bg(x)\} = a\hat{f}(\omega) + b\hat{g}(\omega)$$

DIFFERENTIAL:

$$\boxed{\mathcal{F}\{f'(x)\} = i\omega\hat{f}(\omega)}$$

$$\boxed{\mathcal{F}\{f''(x)\} = -\omega^2\hat{f}(\omega)}$$

Proof:

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x}\underbrace{f'(x)\,dx}_{d f(x)}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\Big[e^{-i\omega x}f(x)\Big]_{-\infty}^{\infty}}_{=0} + i\omega\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx}_{\hat{f}(\omega)}$$

Assume decay: $[f(\pm\infty) = 0]$

$$= i\omega\, \mathcal{F}\{f(x)\} \qquad \square$$

See "Convolution (of Signals)", homepage for a picture.

CONVOLUTION:

DEF. $$\boxed{\underbrace{\mathcal{F}\{f * g\}}_{\text{CONVOLUTION}} = \sqrt{2\pi}\, \underbrace{\mathcal{F}\{f\}\,\mathcal{F}\{g\}}_{\text{MULTIPLICATION}}}$$

DEF. $$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

$$f * g = g * f$$
$$(f * g) * h = f * (g * h)$$

Proof of $\widehat{f * g} = \sqrt{2\pi}\, \hat{f} \cdot \hat{g}$.

$$\mathcal{F}\{(f*g)(x)\} \overset{\text{DEF.}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{-\infty} f(x-y)\, g(y)\, dy\; e^{-i\omega x}\, dx$$

Def. of $*$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \Big(\underbrace{\int_{-\infty}^{\infty} f(x-y)\, e^{-i\omega x}\, dx}_{\sqrt{2\pi}\, e^{i\omega y}\, \hat{f}(\omega)} \Big)\, dy$$

Inner integral $\displaystyle\int_{-\infty}^{\infty} f(x-y)\, e^{-i\omega x}\, dx$

Subst
$z = x - y$
$dz = dx$

$$= \int_{-\infty}^{\infty} f(z)\, e^{-i\omega z}\, e^{-i\omega y}\, dz$$

$$= e^{i\omega y} \sqrt{2\pi}\, \hat{f}(\omega)$$

$$= \int_{-\infty}^{\infty} g(y) e^{-i\omega y}\, dy \cdot \hat{f}(\omega) = \sqrt{2\pi}\, \hat{g}(\omega)\, \hat{f}(\omega) \qquad \square$$

<u>Important:</u> $\widehat{e^{-ax^2}} = ?$ $\quad a > 0$

GAUSSIAN $\quad$ [K.F. Gauß 1777–1855]

The Fourier transform is again a Gaussian.

$$\boxed{\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}}\, e^{-\frac{\omega^2}{4a}}} \qquad (a > 0)$$

The formula is very appealing for $a = \frac{1}{2}$.

Recall $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

See homepage, "Fourier transform of Gaussian distribution."