Ex $\sin(z) = 2 \qquad z = ?$

$$\frac{e^{iz} - e^{-iz}}{2i} = 2, \quad e^{iz} - e^{-iz} - 4i = 0,$$

$$(e^{iz})^2 - 4ie^{iz} - 1 = 0 \qquad \text{[Quadratic eqn.]}$$

$$e^{iz} = 2i \pm \sqrt{(2i)^2 + 1} = 2i \pm \sqrt{-3}$$

$$= i(2 \pm \sqrt{3})$$

$$iz = \ln[i(2 \pm \sqrt{3})] = \ln|i(2 \pm \sqrt{3})| + i\frac{\pi}{2} + 2ni\pi$$

$$\underline{z} = -i\ln|2 \pm \sqrt{3}| + \frac{\pi}{2} + 2n\pi$$

$$= \underline{\mp i \ln(2+\sqrt{3}) + \frac{\pi}{2} + 2n\pi}. \qquad \text{[}\sin(z)\text{ has the period }2\pi.\text{]}$$

#

Remark: $\arcsin(z)$, $\arccos(z)$, $\operatorname{arsinh}(z)$, ... can all be written in terms of logarithms. For example

$$\arctan(z) = \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right).$$

CAUTION:

$$\sin(x)^2 + \cos(x)^2 = 1$$

$$\sin(z)^2 + \cos(z)^2 = 1$$

$$|\sin(z)|^2 + |\cos(z)|^2 = \cdots = \cosh(2y)$$

$$> 1 \text{ if } y \neq 0. \quad (z = x + iy)$$

COMPLEX INTEGRALS $\int_C f(z)\,dz$

along curves

CURVES

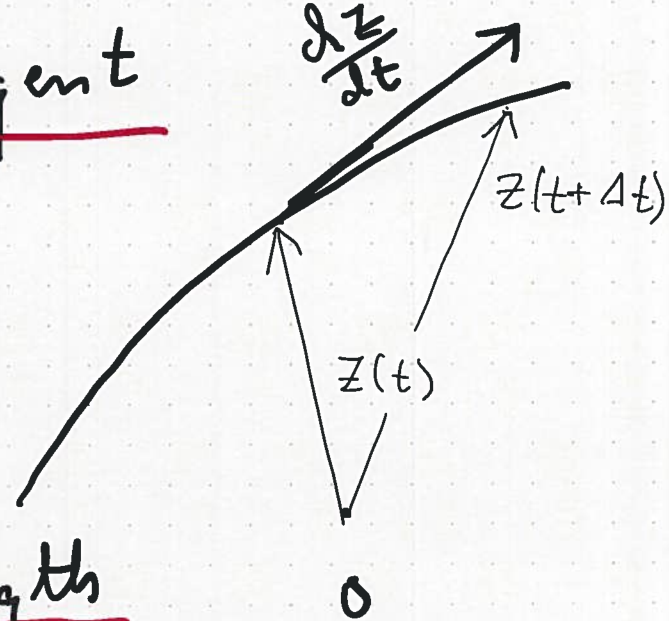
$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \iff \vec{r} = \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

$$\boxed{z(t) = x(t) + i\,y(t), \quad i^2 = -1}$$

$$\dot{z}(t) = \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

$$= \frac{dx}{dt} + i\frac{dy}{dt}$$

tangent

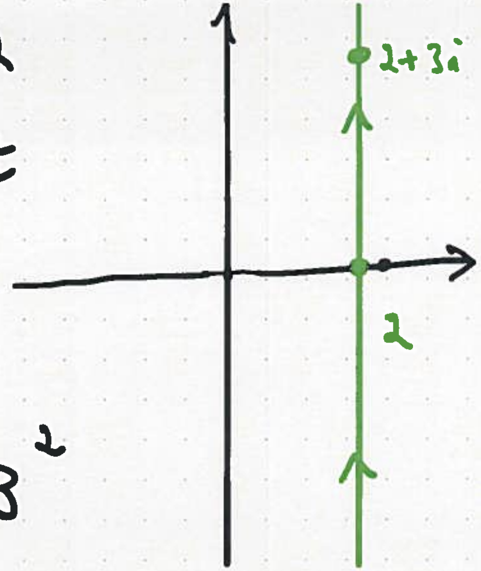


$$s = \int_\alpha^\beta |\dot{z}(t)|\,dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \text{length}$$

Ex. $z(t) = 2 + it \quad (-\infty < t < \infty)$ $\quad \begin{cases} x = 2 \\ y = t \end{cases}$

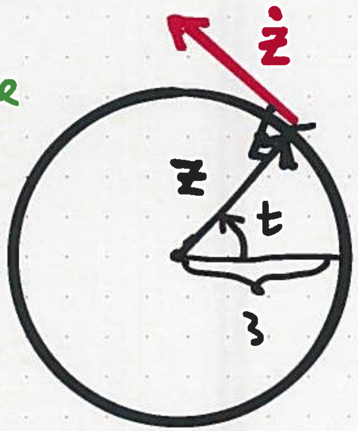
[A line]



$\dot{z}(t) = i$ [tangent]

Ex.: $z = 3e^{it}$ Note $\underbrace{\bar{z} \cdot z}_{|z|^2} = 3^2$

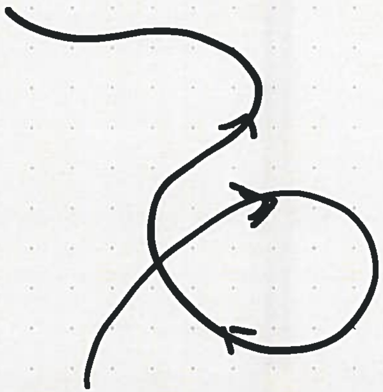
A circle

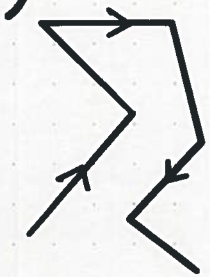


$\dot{z} = 3ie^{it} \quad (= iz)$

$\begin{cases} x = 3\cos(t) \\ y = 3\sin(t) \end{cases}$ (Usually we write $t = \theta$)

Ex

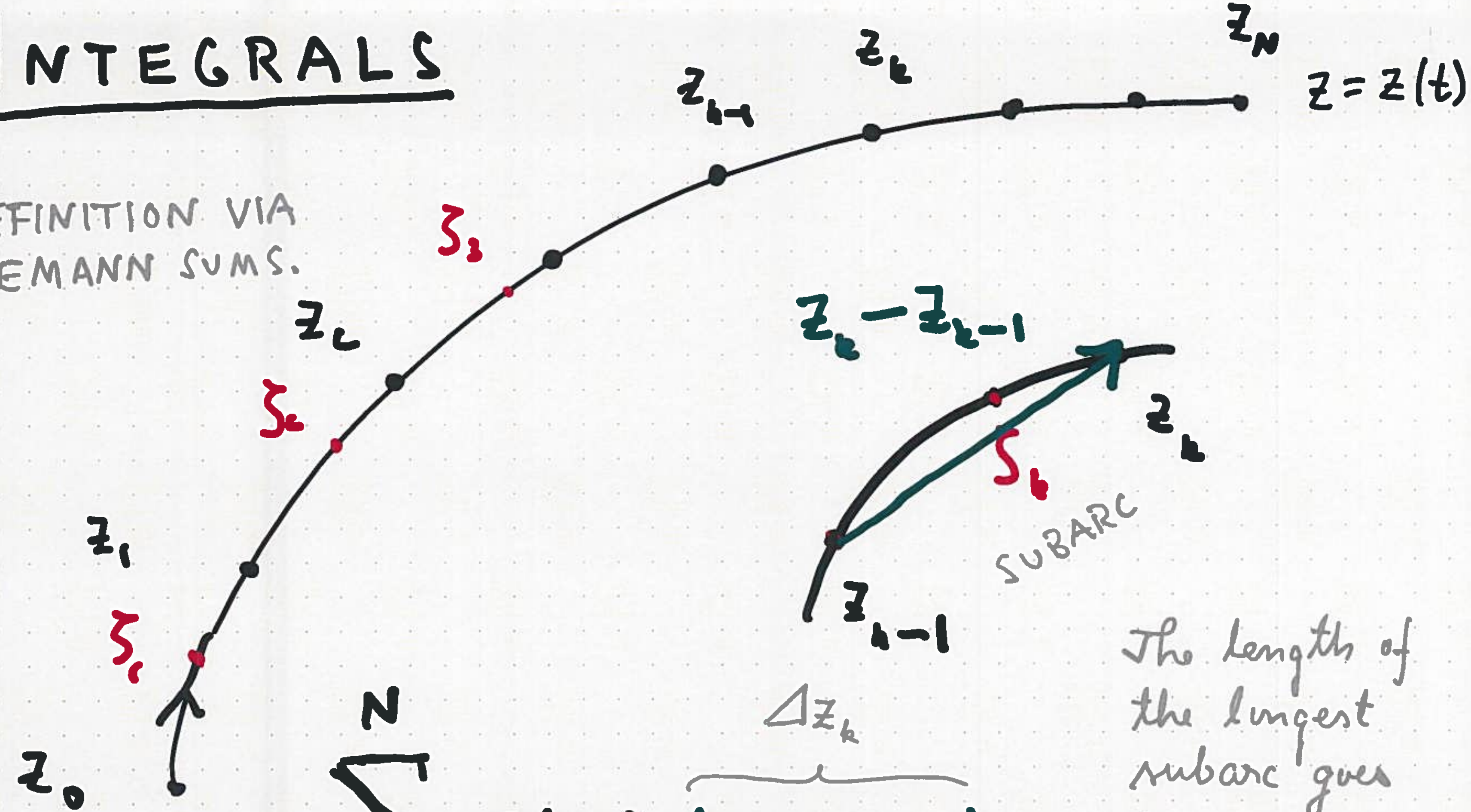




Piecewise smooth curves.

INTEGRALS

DEFINITION VIA RIEMANN SUMS.



The length of the longest subarc goes to zero.

$$\sum_{k=0}^{N} f(\zeta_k)\cdot\underbrace{(z_k - z_{k-1})}_{\Delta z_k} \longrightarrow \int_C f(z)\,dz$$

Piecewise smooth. → C

Continuous on the curve C

How to calculate $\int_C f(z)\,dz$

Method I

$$z(t) = x(t) + i\,y(t)$$

$$dz = dx + i\,dy$$

$$f = u + i\,v$$

$$f(z)\,dz = (u + iv)(dx + i\,dy)$$

$$= u\,dx - v\,dy + i\,(v\,dx + u\,dy)$$

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy)$$

Two real line integrals

NOTE $W = \int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy \quad (\vec{F} = P\hat{i} + Q\hat{j})$

work!

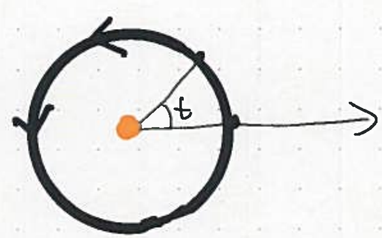
In practical terms:

$$\boxed{\int_C f(z)\,dz = \int_\alpha^\beta f(z(t))\,\underbrace{\dot{z}(t)\,dt}_{dz(t)}}$$

Substitute $z \longleftrightarrow z(t)$!

Ex $\displaystyle\oint_{|z|=1} \frac{dz}{z} = ?$ (Integrate $\frac{1}{z}$ along the circle $z = e^{it}$, $0 \le t \le 2\pi$.)

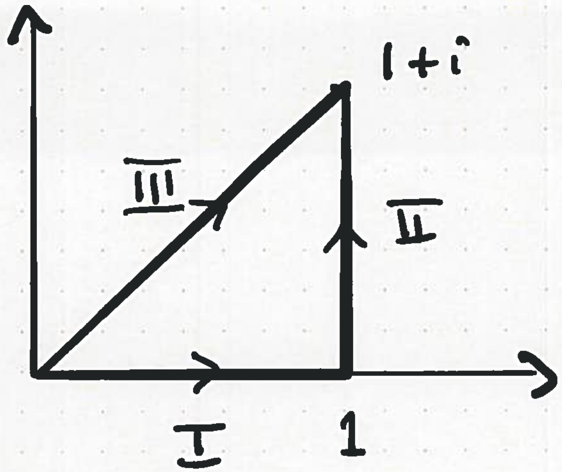
$$\begin{cases} z = e^{it} \\ dz = i e^{it}\,dt \end{cases}$$



$$\oint_C \frac{dz}{z} = \int_{t=0}^{2\pi} \frac{i\cancel{e^{it}}\,dt}{\cancel{e^{it}}} = \underline{i \cdot 2\pi}$$

$$\boxed{\frac{1}{2\pi i}\int_{|z|=1} \frac{dz}{z} = 1}$$

Ex $\displaystyle\int_C \bar{z}\,dz = ?$



1) $\displaystyle\int_{I} \bar{z}\,dz = \int_0^1 x\,dx = \Big/_0^1 \frac{x^2}{2} = \underline{\frac{1}{2}}$

$\begin{cases} z = x \\ dz = dx \end{cases}$

$\bar{z}$ is not analytic.

2) $\displaystyle\int_{II} \bar{z}\,dz = \int_0^1 (1-iy)\,i\,dy = \int_0^1 (i+y)\,dy = \underline{i + \frac{1}{2}}$

$\begin{cases} z = 1 + iy, & 0 \le y \le 1 \\ dz = i\,dy \end{cases}$

3) $\displaystyle\int_{III} \bar{z}\,dz = \int_0^1 (t - it)(1+i)\,dt = 2\int_0^1 t\,dt = \Big/_0^1 t^2 = \underline{1}$

$\begin{cases} z = t + it, & 0 \le t \le 1 \\ dz = (1+i)\,dt \end{cases}$

NOTE $\displaystyle\int_{I} + \int_{II} = \int_{III}$

#

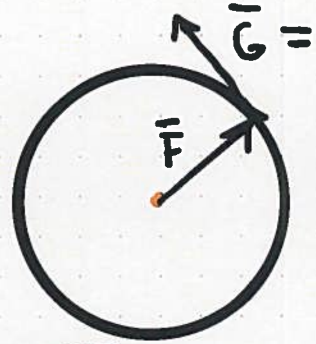
About

$$2\pi i = \oint_{|z|=1} \frac{dz}{z} = \underbrace{\oint x\,dx + y\,dy}_{\int \vec{F}\cdot d\vec{r} \;=0} + i \underbrace{\oint x\,dy - y\,dx}_{2\pi = \int \vec{G}\cdot d\vec{r}}$$

$$\frac{dz}{z} = \frac{dx + i\,dy}{x + iy} = \frac{(x-iy)(dx + i\,dy)}{\underbrace{x^2+y^2}_{=1}}$$

$$= x\,dx + y\,dy + i(x\,dy - y\,dx)$$

The work done by the force $\vec{F}$ is zero.



$\bar{G} = -y\hat{i} + x\hat{j}$

$\bar{F} = x\hat{i} + y\hat{j}$

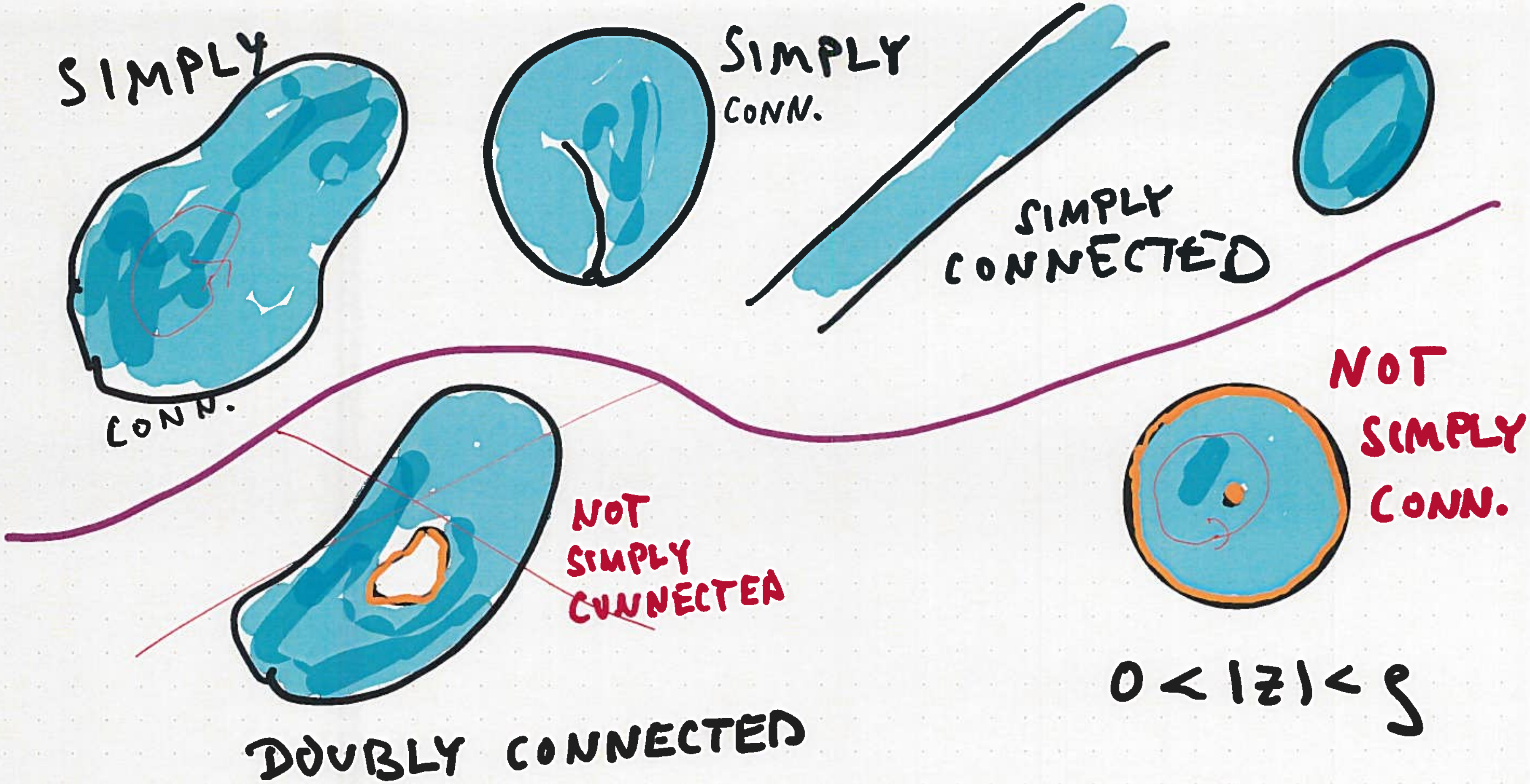
Method II

$$\boxed{\int_a^b f(z)\,dz = F(b) - F(a)}$$

$$\frac{dF}{dz} = f$$

if $F'(z) = f(z)$ in a SIMPLY connected domain containing the curve.

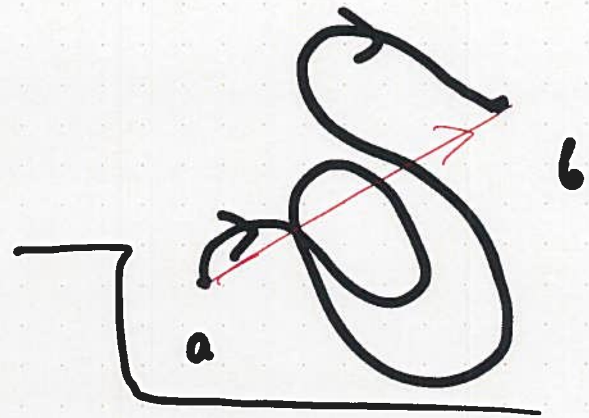
PATH INDEPENDENT!



Ex

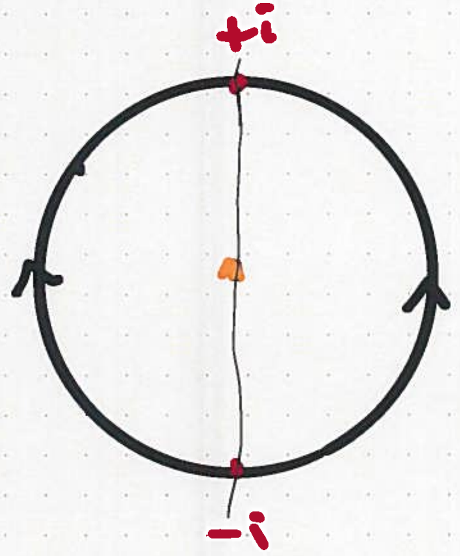
$$\int_a^b z^2\,dz = \Big/_a^b \frac{z^3}{3} = \frac{b^3 - a^3}{3}$$

$$\int_a^b \cos(z)\,dz = \Big/_a^b \sin(z) = \sin(b) - \sin(a).$$



$\underline{\text{Ex:}}$ $\dfrac{d}{dz}\ln(z) = \dfrac{1}{z}$, $z \neq 0$

$\boxed{\ln(z) \text{ is } \underline{\text{not}} \text{ continuous along } |z|=1. \ 2n\pi i\,!}$



$$\int_{-i}^{i} \frac{dz}{z} = \pi i \quad (\text{right half-circle})$$
$x>0$

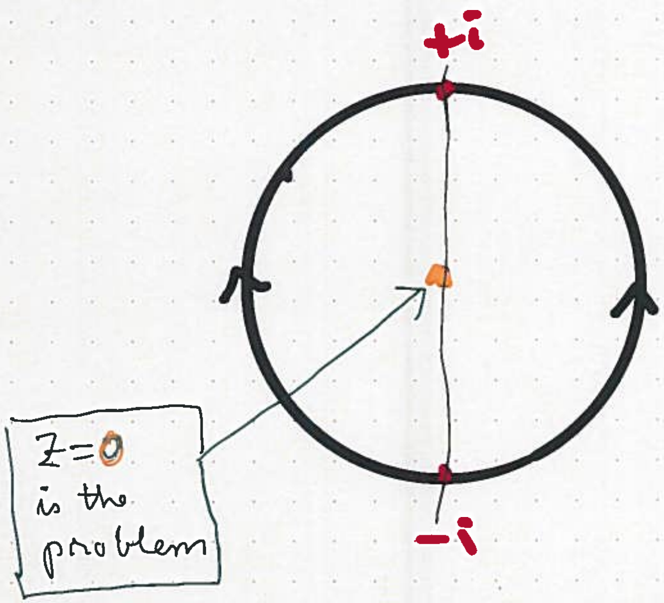
$$\int_{-i}^{i} \frac{dz}{z} = -\pi i \quad (\text{left half-circle})$$
$x<0$

$$\int_{-i}^{i} \frac{dz}{z} = \log(i) - \log(-i)$$

NOT A COMMON SIMPLY CONNECTED DOMAIN.

Ex: $\frac{d}{dz}\ln(z) = \frac{1}{z}$, $z \neq 0$

ln(z) is not continuous along $|z|=1$. $2n\pi i$!



$$\int_{-i}^{i} \frac{dz}{z} = \pi i \quad \text{(right half-circle)}$$

$x>0$

$$\int_{-i}^{i} \frac{dz}{z} = -\pi i \quad \text{(left half-circle)}$$

$x<0$

$$\int_{-i}^{i} \frac{dz}{z} = \log(i) - \log(-i)$$

NOT A COMMON SIMPLY CONNECTED DOMAIN.

for both half-circles.