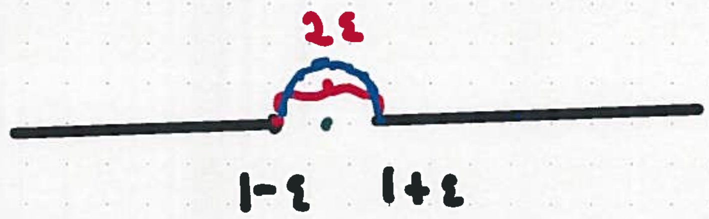
Ex $$PV\int_{-\infty}^{\infty}\frac{dx}{(x-1)(x^2+4)} = \lim_{\varepsilon\to 0}\int_{|x-1|\geq\varepsilon}\frac{dx}{(x-1)(x^2+4)}$$

[The integral diverges, but a P.Value exists]



$$f(z) = \frac{1}{z-1}\cdot\frac{1}{(z-2i)(z+2i)}$$

$$\operatorname*{Res}_{z=1}\{f(z)\} = \frac{1}{(1-2i)(1+2i)} = \frac{1}{5}$$

Simple poles: $1, \pm 2i$

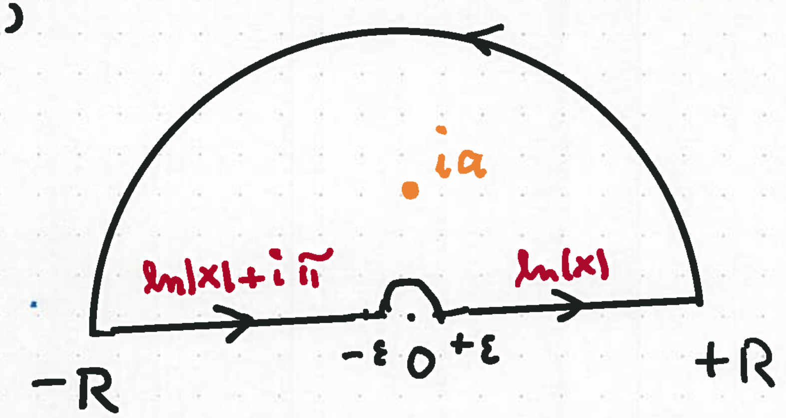
$2i$ is in the upper half-plane.

$$\operatorname*{Res}_{z=2i}\{f(z)\} = \frac{1}{2i-1}\,\frac{1}{(2i+2i)} = \frac{1}{4i(2i-1)}$$

$$PV\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\operatorname*{Res}_{z=2i}\{f(z)\} + \underset{50\%}{\pi i}\operatorname*{Res}_{z=1}\{f(z)\} = -\frac{\pi}{10}.$$

Ex

$$\int_0^\infty \frac{\ln(x)}{x^2+a^2}\,dx = ? \quad , \quad a > 0.$$



$$f(z) = \frac{\ln(z)}{z^2+a^2}$$

Simple poles at $z = \pm ia$

$$\boxed{\begin{array}{c}\ln(re^{i\theta}) = \ln(r) + i\theta \\ 0 \le \theta \le \pi\end{array}}$$

$$2\pi i \operatorname*{Res}_{z=ia}\{f(z)\} = \oint \frac{\ln(z)\,dz}{z^2+a^2}$$

$$= \int_{-R}^{-\varepsilon} \frac{\ln|x|+i\pi}{x^2+a^2}\,dx + \int_{\varepsilon}^{R} \frac{\ln(x)\,dx}{x^2+a^2} +$$

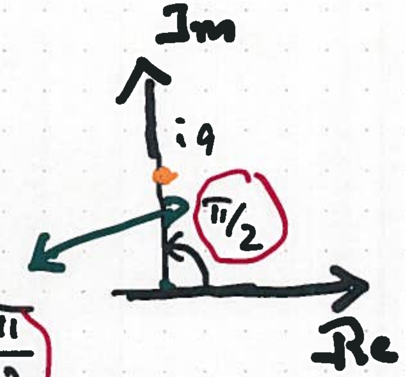
$z = \varepsilon e^{i\theta}$ $\quad dz = i\varepsilon e^{i\theta}d\theta$

$z = Re^{i\theta}$
$dz = iRe^{i\theta}d\theta$

$$+\underbrace{\int_{\pi}^{0}\frac{\ln(\varepsilon)+i\theta}{\varepsilon^2 e^{2i\theta}+a^2}\, i\varepsilon e^{i\theta}}_{\text{Small Half-circle}} + \underbrace{\int_{0}^{\pi}\frac{\ln(R)+i\theta}{R^2 e^{2i\theta}+a^2}\, iRe^{i\theta}\, d\theta}_{\text{Big half-circle}}$$

$\longrightarrow 0$ as $\varepsilon \to 0+$ $\qquad$ $\longrightarrow 0$ as $R \longrightarrow \infty$

$$\left|\frac{(\ln(\varepsilon)+i\theta)e^{i\theta}\varepsilon i}{\varepsilon^2 e^{2i\theta}+a^2}\right| \le \frac{(\pi+\ln\frac{1}{\varepsilon})\,\varepsilon}{a^2-\varepsilon^2} \underset{\text{as } \varepsilon\to 0}{\longrightarrow} 0.$$



$$\operatorname*{Res}_{z=ia}\left[\frac{\ln(z)}{(z-ia)(z+ia)}\right] = \frac{\ln(ia)}{ia+ia} = \frac{\ln(a)+i\frac{\pi}{2}}{2ia}$$

Hence

$$\cancel{2\pi i}\left[\frac{\ln(a)+i\frac{\pi}{2}}{\cancel{2i}a}\right] = 2\int_0^\infty \frac{\ln(x)}{x^2+a^2}\,dx + i\int_{\cancel{-\infty}\ 0}^{\cancel{\infty}\ \infty}\frac{\pi}{x^2+a^2}\,dx$$

Answer: $$\int_0^\infty \frac{\ln(x)}{x^2+a^2}\,dx = \frac{\pi\ln(a)}{2a}.$$

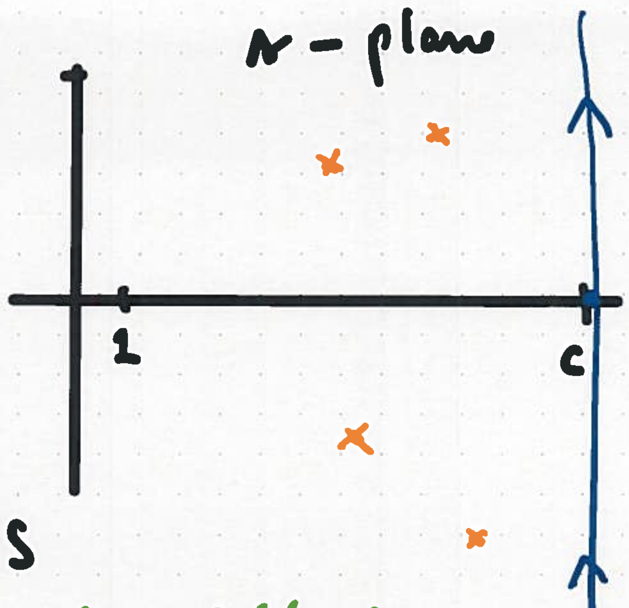
Also
$$\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$$
(Imaginary part).

LAPLACE TRANSF*

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{+st}\,ds$$

[Not in the syllabus.]



The real variable s is replaced by a complex one.

$s = c + iy, \; -\infty < y < \infty$

$c >> 1$