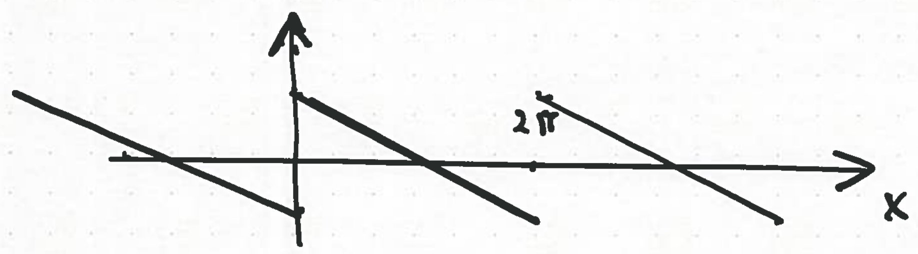
Ex.: $$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}, \quad \underline{0 < x < 2\pi}$$



$$\begin{cases} a_0 = 0 \\ a_n = \frac{1}{n}, \quad n \geq 1 \\ b_n = 0 \end{cases}$$

Parseval: $$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = a_0^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \cdots = \frac{\pi^2}{6}. \quad \#$$

Ex. $f(x) = x^2, \quad -\pi \le x \le \pi.$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}, \quad |x| \le \pi$$

How many terms are needed to make the mean square error $E_N < 10^{-6}$ E_N

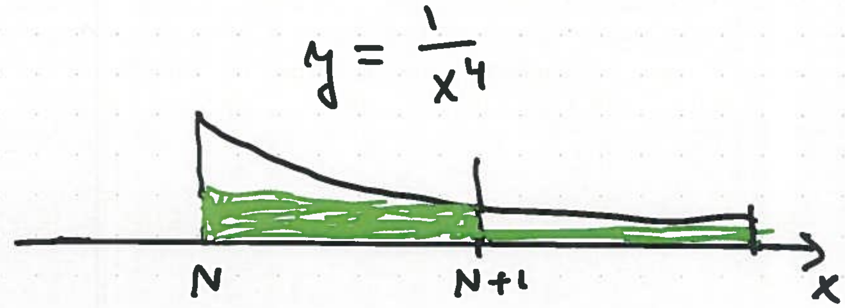
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(x^2 - \frac{\pi^2}{3} - 4\sum_{n=1}^{N}\frac{(-1)^n\cos(nx)}{n^2}\right)^2 dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx - \left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{N}\frac{16}{n^4}\right],$$

$$= \sum_{n=N+1}^{\infty} \frac{8}{n^4} \overset{?}{<} 10^{-6}$$

Compare areas!

$$< \int_N^{\infty} \frac{8}{x^4}\,dx = \Big|_N^{\infty} -\frac{8}{3x^3}$$



$$= \frac{8}{3N^3}$$

$N = 140$ OK.

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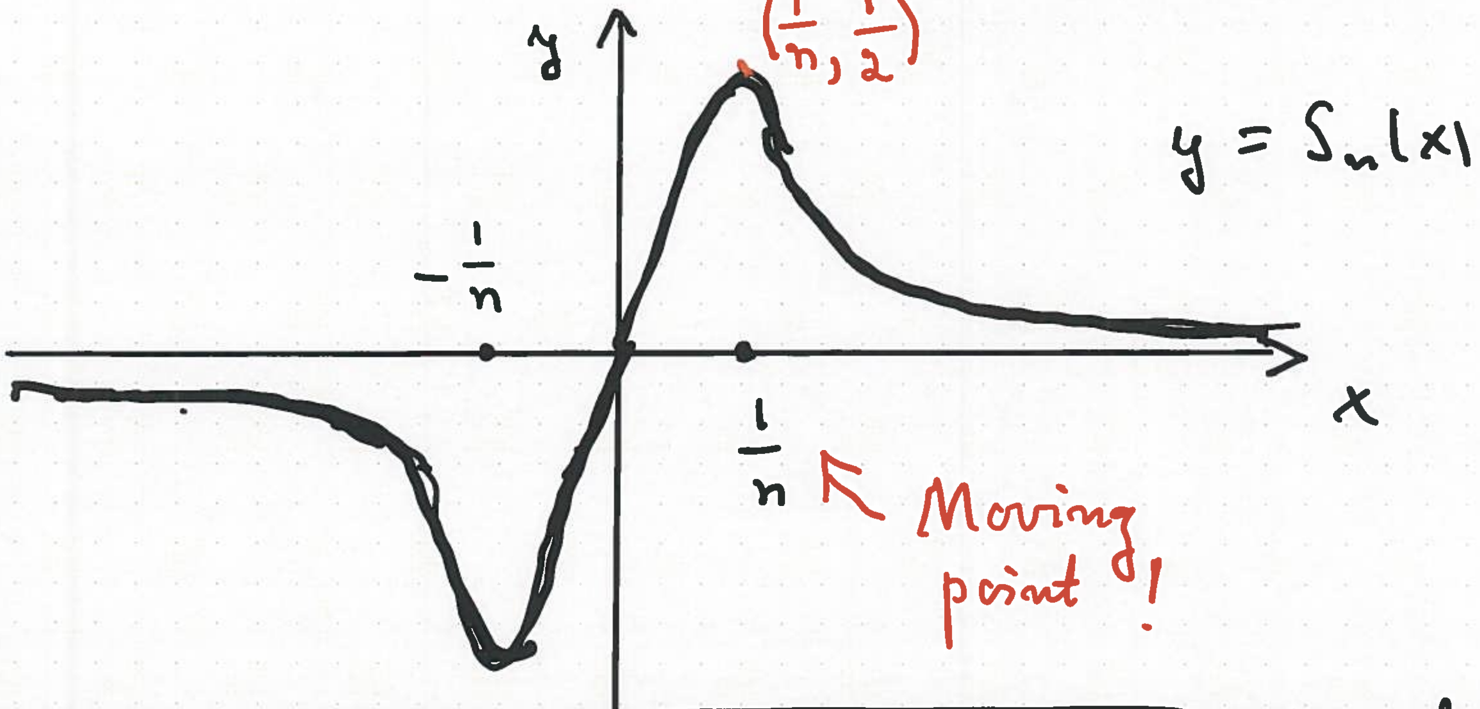
Remark: $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. (from Parseval)

About Gibbs' Phenomenon

$$S_n(x) = \frac{nx}{1+n^2x^2}, \quad -\infty < x < \infty$$

$n = 1, 2, 3, 4, \ldots$



$$\boxed{S(x) = \lim_{n\to\infty} S_n(x) = 0}$$

pointwise

At each **fixed** point.

Indeed, $\qquad S_n(0) = 0$

$\underline{x=0}$ $\quad S(0) = \lim_{n\to\infty} S_n(0) = 0$

$\underline{x \neq 0}$ $\quad |S_n(x)| = \left| \dfrac{nx}{1+n^2x^2} \right| \leq \dfrac{|nx|}{n^2x^2}$.

$$= \frac{1}{n|x|} \longrightarrow 0 \text{ as } n \to \infty$$

<u>Notice</u> Given $\varepsilon > 0$, $x \neq 0$

$$|S_n(x) - S(x)| < \frac{1}{n|x|} < \varepsilon$$

when $n > n_\varepsilon(x) = \dfrac{1}{\varepsilon|x|}$

The index depends on the point x. Bad for $x \sim 0$.

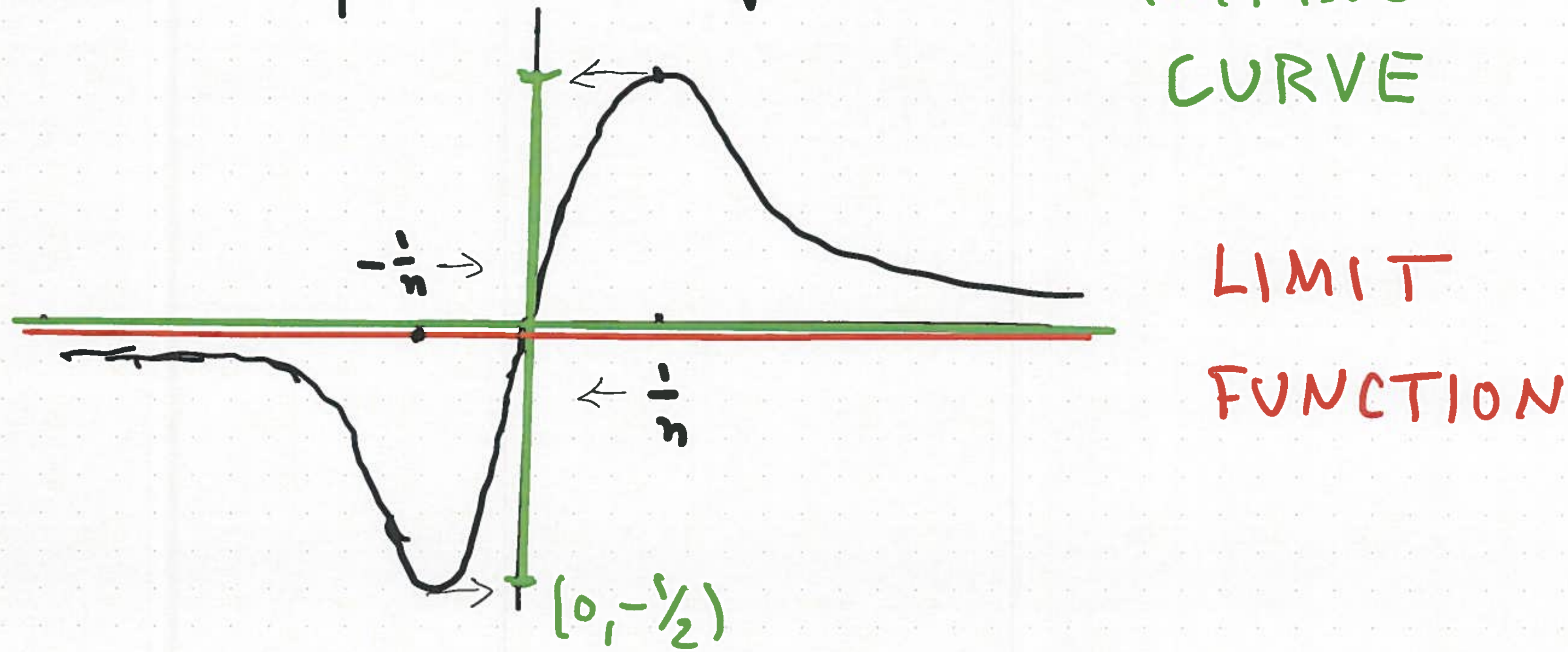
! $\quad S_n\left(\frac{1}{n}\right) = \frac{1}{2}$!

$(n = 1, 2, 3, 4, \ldots)$

Now $\max\limits_{x} |S_n(x) - S(x)| = |S_n(\tfrac{1}{n}) - 0|$

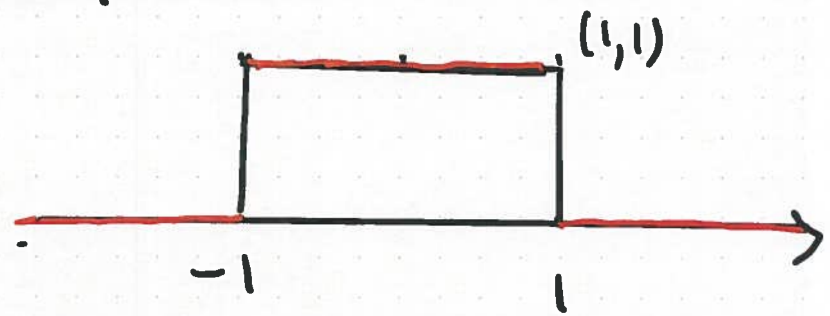
$$= \frac{1}{2} \not\longrightarrow 0$$

Non-uniform convergence!



#

Ex $f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & |x| > 1 \end{cases}$

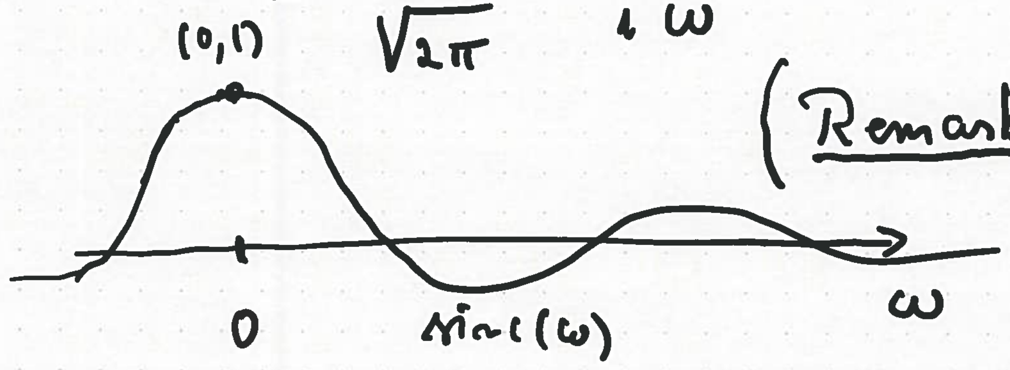


$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \, e^{-i\omega x} \, dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-i\omega x}}{-i\omega} \right|_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega} - e^{-i\omega}}{i\omega} = \sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega}$$

(**Remark:** $\operatorname{sinc}(\omega) = \frac{\sin(\omega)}{\omega}$.)



$$\begin{cases} \hat{f}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \\[2ex] f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{+i\omega x}\, d\omega \end{cases}$$

FOURIER TRANFORM

Measures how much of the frequency ω is present in the "signal" $f(x)$

NOTATION

$$\hat{f}(\omega) = \mathcal{F}\{f(x)\}$$

$$f(x) = \mathcal{F}^{-1}\{\hat{f}(\omega)\}$$ INVERSE TRANSF.

THE FOURIER TRANSFORM

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx$$

period $= 2L$

$$f(x) = \sum_{n=-\infty}^{\infty} \underline{\underline{c_n}}\, e^{+i\frac{n\pi x}{L}}$$

What happens as $L \to \infty$.

c_n measures how much of the function oscillates at frequency $= \frac{n\pi}{L}$.

If the procedure $L \to \infty$ is <u>very</u> controlled, we obtain:

Inverse

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin(\omega)}{\omega} e^{+i\omega x} \, d\omega = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \\ \left(\frac{1}{2}, \right. & \left. x = \pm 1 \right) \end{cases}$$

$x = 0$ $$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\omega)}{\omega} \, d\omega = 1$$

We know already that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(t)}{t} \, dt = \pi.$

EULER

RECALL $$e^{\pm i\omega x} = \cos(\omega x) \pm i \sin(\omega x)$$

$$|e^{\pm i\omega x}| = 1 \quad \text{(bounded function)}$$

ADDENDUM: PARSEVAL $$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2$$