Ex. (!) $\dfrac{1+e^{i\pi z}}{(z-1)^2(z+1)^2}$ Find the singularities and characterize them.

Obviously $z = \pm 1$ are poles (or removable).
They are SIMPLE poles. $\left(1+e^{i\pi(\pm 1)} = 0\right)$.

$$1+e^{i\pi z} = 1+e^{i\pi(z-1)}e^{i\pi} = 1-e^{i\pi(z-1)}$$
$$= 1-\left(1+i\pi(z-1)+\frac{(i\pi)^2}{2!}(z-1)^2+\cdots\right)$$
$$= -(z-1)\left[i\pi+\frac{(i\pi)^2}{2!}(z-1)+\cdots\right]$$

Thus $z-1$ cancels. The pole $z=1$ is simple.
The same for $z=-1$. ∎

RATIONAL INTEGRALS

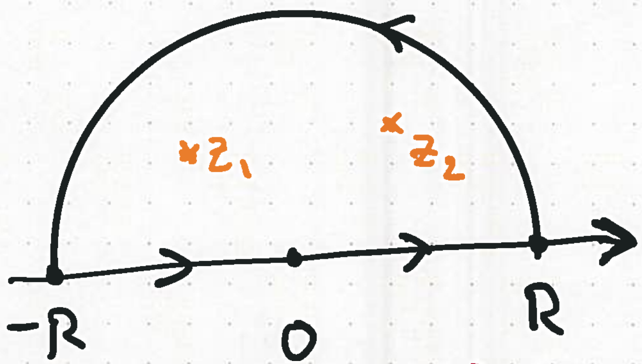
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \underbrace{2\pi i \sum_{y_k>0} \operatorname*{Res}_{z=z_k}\left\{\frac{P(z)}{Q(z)}\right\}}_{\text{Sum of the residues in the upper half-plane.}}$$

$z_k = x_k + i y_k$

↑ Polynomials

1°) $Q(x) \neq 0$ when $-\infty < x < +\infty$
[No real zeros]

2°) $\deg(Q) \geq 2 + \deg(P)$

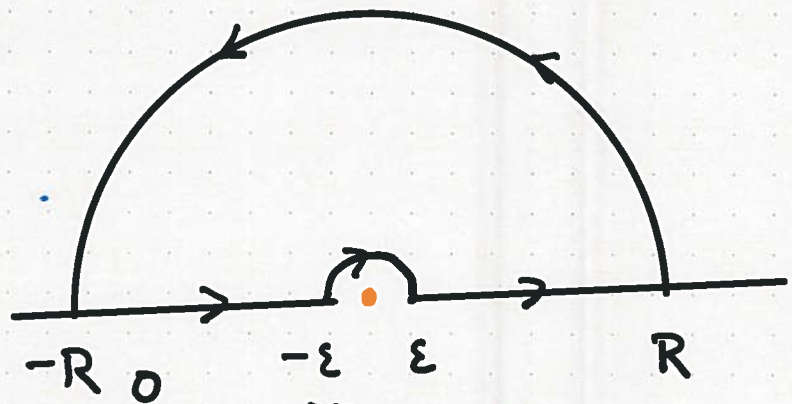


The half-circle integral goes to zero as $R \to \infty$, (by 2°)

Ex: $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \operatorname*{Res}_{z=i}\left\{\frac{1}{1+z^2}\right\} = \cdots = \pi.$$

Ex.: $\displaystyle\int_0^\infty \frac{\sin(x)}{x}\,dx = ?$ Start with

Answer: $\frac{\pi}{2}$



$$0 = \oint_{\Omega} \frac{e^{iz}}{z}\,dz$$

$$= \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\cancel{\varepsilon e^{i\theta}}}\, i\cancel{\varepsilon e^{i\theta}}\,d\theta$$

$$+ \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{\cancel{Re^{i\theta}}}\, i\cancel{Re^{i\theta}}\,d\theta$$

$$\left|e^{iRe^{i\theta}}\right| = e^{-R\sin(\theta)}$$

$$\left|\int_0^\pi\right| \le \pi \int_0^\pi e^{-R\sin(\theta)}\,d\theta \xrightarrow[R\to\infty]{} 0.$$

$R \to \infty$

$$= \int_{|x|\ge\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\pi}^{0} e^{i\varepsilon e^{i\theta}}\, i\,d\theta + 0$$

$$i\int_{\pi}^{0} e^{i\varepsilon e^{i\theta}}\, d\theta \xrightarrow{\varepsilon\to 0^+} -\pi i$$

$$\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{e^{ix}}{x}\, dx = \pi i$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi}$$

$$PV\int_{-\infty}^{\infty} \frac{\cos(x)}{x}\, dx$$

$$= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{\cos(x)}{x}\, dx = 0 .$$

Hence

$$e^{ix} = \cos(x) + i\sin(x)$$

$$\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{\cos(x)}{x}\, dx = 0$$

Comments

$$\int_{0}^{\infty} \frac{\cos x}{x}\, dx = +\infty$$

$$\int_{-\infty}^{0} \frac{\cos(x)}{x}\, dx = -\infty$$

"$\infty - \infty = 0$"

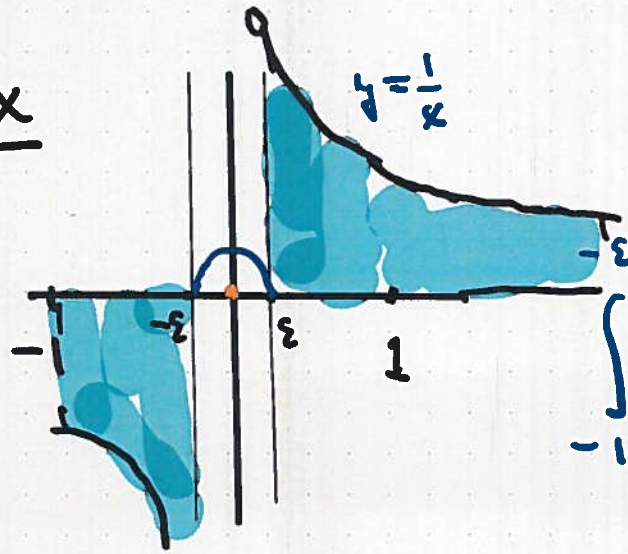
PRINCIPAL VALUES / SIMPLE POLES ON THE CONTOUR.

So far, the poles were not allowed to be on the curve (contour) of integration. What happens if they do?

Ex $\int_{-1}^{1} \frac{dx}{x^2} = \Big|_{-1}^{1} -\frac{1}{x} = -2$ TOTALLY WRONG

Now $\int^{1} \frac{dx}{x^2} = +\infty, \quad \int_{-1}^{0} \frac{dx}{x^2} = +\infty$

Ex



$\int_{-1}^{2} \frac{dx}{x} = ?$ ISOLATE $(-\varepsilon, \varepsilon)$.

$\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{2} \frac{dx}{x} = \Big|_{-1}^{-\varepsilon} \ln|x| + \Big|_{\varepsilon}^{2} \ln|x| = \ln(2)$

"$\infty - \infty = \ln(2)$" Cancellation effect.

We say that

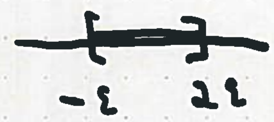
$$PV\int_{-1}^{2}\frac{dx}{x} = \lim_{\varepsilon\to 0}\left[\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{2}\right] = \ln(2)$$

THIS IS THE PRINCIPAL VALUE OF THE INTEGRAL.

Notice $\displaystyle\int_{-1}^{0}\frac{dx}{x} = -\infty\,,\quad \int_{0}^{2}\frac{dx}{x} = +\infty$ $\qquad \infty - \infty = \ln(2)$

CAUTION

$$\left(\int_{-1}^{-\varepsilon} + \int_{2\varepsilon}^{2}\right)\frac{dx}{x} = \ln|-\varepsilon| + \ln(2) - \ln(2\varepsilon)$$
$$= \ln\left[\frac{\varepsilon}{2\varepsilon}\right] + \ln(2) = 0$$

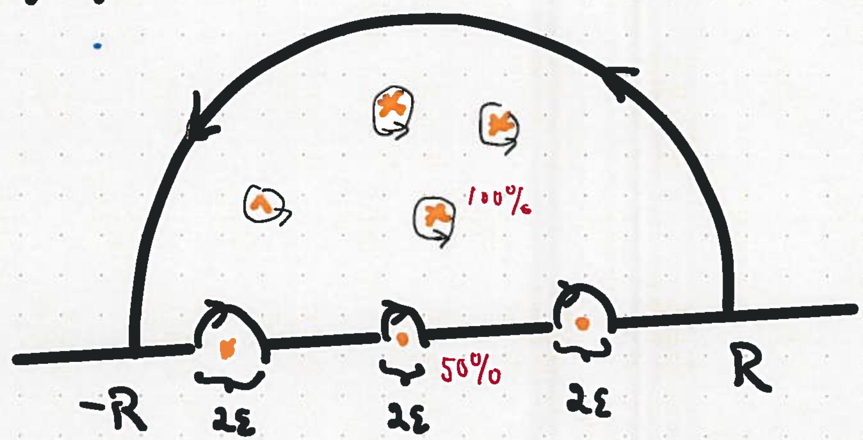


Here the isolation of 0 was not the symmetric $(-\varepsilon, \varepsilon)$ but the biased $(-\varepsilon, 2\varepsilon)$. Sensitive!

$$PV\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx = 2\pi i\underbrace{\sum_{y_k>0}\overset{100\%}{\operatorname{Res}_{z=z_k}}\frac{P(z)}{Q(z)}}_{\text{The sum of the residues in the upper half-plane}} + \pi i\underbrace{\sum_{x_k}\overset{50\%}{\operatorname{Res}_{z=x_k}}\frac{P(z)}{Q(z)}}_{\text{The sum of the residues on the real axis.}}$$

1°) $Q(x)\neq 0$ except at SIMPLE zeros on the real axis.

2°) $\deg Q \geq 2+\deg P.$

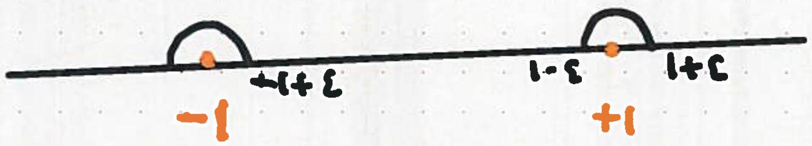


Ex.:

$$PV\int_{-\infty}^{\infty}\frac{dx}{x^2-1} \overset{\text{DEF}}{=} \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right\}\frac{dx}{x^2-1}$$

$x=\pm 1$

The integral is divergent, but a principal value can be assigned to it.



$$= \overset{100\%}{2\pi i \cdot 0} + \overset{50\%}{\pi i} \left[\operatorname*{Res}_{z=-1} \frac{1}{z^2-1} + \operatorname*{Res}_{z=1} \frac{1}{z^2-1} \right]$$

$$\operatorname*{Res}_{z=1} \frac{1}{z^2-1} = \lim_{z \to 1} \frac{z-1}{(z-1)(z+1)} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=-1} \frac{1}{z^2-1} = \lim_{z \to -1} \frac{z+1}{(z-1)(z+1)} = -\frac{1}{2}$$

$$= 0.$$

Answer $= 0$.

REMARK Integrals

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\omega x)\, dx, \quad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\omega x)\, dx$$

Subject: Complex Analysis
From: Peter Lindqvist <peter.lindqvist@ntnu.no>
Date: 13/11/2020, 11:10
To: Peter Lindqvist <peter.lindqvist@ntnu.no>

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$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\omega x}\, dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\omega x)\, dx + i \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\omega x)\, dx$$

Assume again 1°) and 2°),

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\omega x}\, dx = \begin{cases} 2\pi i \sum\limits_{y_k>0} \operatorname{Res}\left\{\frac{P(z)}{Q(z)} e^{i\omega z}\right\} & \text{if } \omega > 0 \\ -2\pi i \sum\limits_{y_k<0} \operatorname{Res}\left\{\frac{P(z)}{Q(z)} e^{i\omega z}\right\}, & \text{if } \omega < 0 \end{cases}$$

The residues in the upper half plane.

Residues in the lower half-plane.

$z_k = x_k + i y_k$

1°) $Q(x) \neq 0$

2°) $\deg Q \geq 2 + \deg P$.

Why 2 formulas: $|e^{i\omega z}| = |e^{i\omega x} e^{-\omega y}| = e^{-\omega y}$

We must have $\omega y > 0$ on the half-circle

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