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Examination paper for **TMA4120 Calculus 4K**

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Problem 1 Let

$$g(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & 2 < t. \end{cases}$$

a) Find the Laplace transform of g.

Løsning By difinition:

$$\mathcal{L}(g) = \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt = \Big(-\frac{1}{s}e^{-st}\Big)\Big|_0^1 + \Big(-\frac{1}{s}e^{-st}\Big)\Big|_1^2.$$

Thus $\underline{\mathcal{L}(g) = s^{-1}(1 - 2e^{-s} - e^{-2s})}$.

Alternatively, $g(t) = u(t) - 2u(t-1) + u(t-2)$, using the tabel, we obtain

$$\mathcal{L}(g) = \mathcal{L}(u(t)) - 2\mathcal{L}(u(t-1)) + \mathcal{L}(u(t-2)) = \frac{1 - 2e^{-s} + e^{-2s}}{s}.$$

b) Solve the initial value problem

$$y''(t) + 4y(t) = g(t), \quad t \ge 0, \ y(0) = 0, \ y'(0) = 0.$$

Løsning We apply the Laplace transform, let $Y = \mathcal{L}(y)$, then using the initial data and computation from part a), we obtain

$$(s^2 + 4)Y = \frac{1 - 2e^{-s} + e^{-2s}}{s}, \quad Y(s) = \frac{1 - 2e^{-2} + e^{-2s}}{s(s^2 + 4)}.$$

We use partial fraction decomposition,

$$Y(s) = (1 - 2e^{-s} + e^{-2s})\Big(\frac{1}{4s} - \frac{s}{s^2 + 4}\Big).$$

Then we apply the inverse Laplace transform

$$\underline{y(t) = \frac{1 + \cos 2t}{4}u(t) - \frac{1 + \cos 2(t-1)}{2}u(t-1) + \frac{1 + \cos 2(t-2)}{4}u(t-2).}$$

Problem 2 Consider the boundary value problem:

$$\begin{cases} u_t = u_{xx}, \ t > 0, \ 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

a) Find all solutions on the form $u(x,t) = F(t)G(x)$ satisfying the end point conditions $G(0) = G(1) = 0$.

Løsning We divide the variables in the equation:

$$\frac{F'}{F} = \frac{G''}{G} = k.$$

Taking into account the boundary condition, we want to find $G(x)$ such that $G'' = kG$ and $G(0) = G(1) = 0$. We know that it is possible only when $k < 0$ and $k = -p^2$. Then $G(x) = \sin px$ and the condition $G(1) = 0$ implies $p = \pi n$ for a positive integer n.

Then $F' = -(\pi n)^2 F$ and $F(t) = Ce^{-(\pi n)^2 t}$. all solutions with divided variables are of the form

$$\underline{u_n(x,t) = C_n \sin(\pi n x) e^{-(\pi n)^2 t}}.$$

b) Use the principle of superposition to find the solution of the boundary value problem that also satisfies the following initial condition

$$u(x,0) = 3\sin(\pi x) + 5\sin(4\pi x), \quad 0 < x < 1.$$

Løsning We know that a linear combination of solutions is a solution (superposition principle), to satisfy the the initial condition we take

$$\underline{u(x,t) = 3\sin(\pi x)e^{-\pi^2 t} + 5\sin(4\pi x)e^{-(4\pi)^2 t}}.$$

Problem 3 Calculate the Fourier transform of the function

$$f(x) = \begin{cases} e^{-2018x}, \ x \geq 0 \\ 0, \quad x < 0. \end{cases}$$

Løsning By the definition

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-2018x - ixw} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{2018 + iw} = \frac{1}{\sqrt{2\pi}} \frac{2018 - iw}{2018^2 + w^2}.$$

Problem 4 Assume that $f(z)$ is an analytic function in a domain Ω that satisfies $|f(z)| = 1$ for all $z \in \Omega$. Prove that f is a constant.

Løsning Consider the function $h(z) = \ln f(z)$, since f has no zeros in Ω h is defined and analytic in Ω. Let $h(z) = u(z) + iv(z)$, where u and v are the real and imaginary parts of h. We have

$$e^{u(z)}e^{iv(z)} = f(z).$$

Taking the absolute values we get $e^{u(z)} = |f(z)| = 1$. It implies that $u(z) = 0$ for all $z \in \Omega$. Then from the Cauchy-Riemann equation we see that $v_x = v_y = 0$ and v is also a constant. Finally we conclude that $f(z) = e^{h(z)}$ is a constant.

Problem 5 Determine the radius of convergence of the Taylor series of the function

$$h(z) = \frac{2}{1 + \cosh(z)}$$

centered at the origin. Explain your answer.

Løsing $h(z)$ is the ratio of two analytic functions $f(z) = 2$ and $g(z) = 1 + \cosh z$. The second function, $g(z)$ is not zero at the origin, so h is analytic at the origin, the Taylor series of h at the origin converges in the largest disk centered at the origin where h is analytic. To determine the radius of this disk we should calculate the distance from the origin to the closest singular point of h. The singular points of h are the zeros of $g(z)$. We solve

$$1 + \cosh(z) = 0 \;\Leftrightarrow\; e^z = -1.$$

let $z = x + iy$ then $e^z = e^x(\cos y + i \sin y)$, if $e^z = -1$ then $x = 0$ and $y = \pi + 2\pi k$ where k is integer. The closest zeros of g to the origin are points $i\pi$ and $-i\pi$. Thus the raduis of convergence of the Taylor series centered at the origin is $\underline{\pi}$.

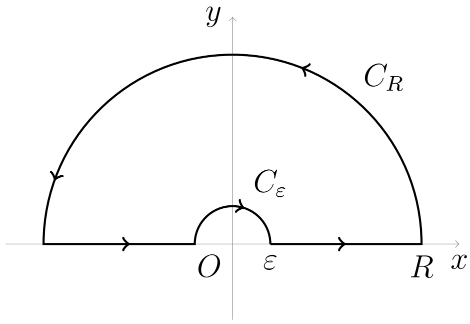
Problem 6 Consider the function

$$f(z) = \frac{\ln(z)}{z^2 + a^2}, \quad a > 0,$$

in the upper half plane. For $z = re^{i\theta}, 0 \le \theta \le \pi$ we define

$$\ln(z) = \ln(r) + i\theta.$$

Let C be the following contour:



a) Determine $\oint_C f(z)\, dz$ by residue calculations.

Løsning First we find the singular points of f in the upper half plane. There is only one, it is $z_0 = ia$. If $R < a$ and the contour does not contain the singular point the the integral is zero. If $R > a$ then by the residue theorem

$$\oint_C f(z)\, dz = 2\pi i \mathrm{Res}_{ia} f(z).$$

Since $f(z)$ has a simple pole at ia, we have

$$\mathrm{Res}_{ia} f(z) = \frac{\ln(ia)}{ia + ia} = \frac{\ln a + i\pi/2}{2ia}.$$

We obtain

$$\oint_C f(z) dz = \frac{\pi(2\ln a + i\pi)}{2a}.$$

b) Show that the integrals over the half circles C_R and C_ε approach zero as $R \to \infty$ and $\varepsilon \to 0$.

Løsning First consider the integral over C_R. We have for $|z| = R$

$$\left| \frac{\ln(z)}{z^2 + a^2} \right| \leq \frac{\ln(R) + \pi}{R^2 - a^2}.$$

Further, integrating over the half-circle of radius R we get

$$\left| \int_{C_r} f(z) dz \right| \leq \frac{\pi R(\ln(R) + \pi)}{R^2 - a^2} \to 0 \quad R \to \infty.$$

For small ε we see that when $|z| = \varepsilon$

$$\left| \frac{\ln(z)}{z^2 + a^2} \right| \leq \frac{|\ln(\varepsilon)| + \pi}{a^2 - \varepsilon^2}$$

and the integral is bounded by

$$\frac{\pi\varepsilon(|\ln \varepsilon| + \pi)}{(a^2 - \varepsilon^2)} \to \quad \varepsilon \to 0.$$

c) Compute
$$\int_0^\infty \frac{\ln(x)\,dx}{x^2+a^2}.$$

Løsning We have

$$\int_{-\infty}^0 \frac{\ln x\,dx}{x^2+a^2} + \int_0^\infty \frac{\ln x\,dx}{x^2+a^2} = \lim_{R\to\infty,\varepsilon\to 0} \oint_C f(z)dz = \frac{\pi(2\ln a + i\pi)}{2a}.$$

On the other hand, for $x < 0$ we have $\ln x = \ln|x| + i\pi$, thus

$$\int_0^\infty \frac{\ln x}{x^2+a^2} = \frac{1}{2}\mathrm{Re}\left(\frac{\pi(2\ln a + i\pi)}{2a}\right) = \frac{\pi\ln a}{2a}.$$