Examination paper for TMA4120 Calculus 4K

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Problem 1  
Let  
\[ g(t) = \begin{cases} 
1, & 0 \leq t < 1 \\
-1, & 1 \leq t < 2 \\
0, & 2 < t. 
\end{cases} \]

a) Find the Laplace transform of \( g \).

\( L(g) \) by definition:
\[ L(g) = \int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt = \left( -\frac{1}{s}e^{-st} \right) \bigg|_0^1 + \left( -\frac{1}{s}e^{-st} \right) \bigg|_1^2. \]
Thus \( L(g) = s^{-1}(1 - 2e^{-s} - e^{-2s}) \).
Alternatively, \( g(t) = u(t) - 2u(t-1) + u(t-2) \), using the table, we obtain
\[ L(g) = L(u(t)) - 2L(u(t-1)) + L(u(t-2)) = \frac{1 - 2e^{-s} + e^{-2s}}{s}. \]

b) Solve the initial value problem
\[ y''(t) + 4y(t) = g(t), \quad t \geq 0, \quad y(0) = 0, \quad y'(0) = 0. \]

\( L(y) \) we apply the Laplace transform, let \( Y = L(y) \), then using the initial data and computation from part a), we obtain
\[ (s^2 + 4)Y = \frac{1 - 2e^{-s} + e^{-2s}}{s}, \quad Y(s) = \frac{1 - 2e^{-2} + e^{-2s}}{s(s^2 + 4)}. \]
We use partial fraction decomposition,
\[ Y(s) = (1 - 2e^{-s} + e^{-2s}) \left( \frac{1}{4s} - \frac{s}{s^2 + 4} \right). \]
Then we apply the inverse Laplace transform
\[ y(t) = \frac{1 + \cos 2t}{4} u(t) - \frac{1 + \cos 2(t-1)}{2} u(t-1) + \frac{1 + \cos 2(t-2)}{4} u(t-2). \]
Problem 2  Consider the boundary value problem:
\[
\begin{aligned}
    u_t &= u_{xx}, \quad t > 0, \quad 0 < x < 1 \\
    u(0, t) &= u(1, t) = 0
\end{aligned}
\]

(a) Find all solutions on the form \( u(x, t) = F(t)G(x) \) satisfying the end point conditions \( G(0) = G(1) = 0 \).

\textit{Løsning} We divide the variables in the equation:
\[
\frac{F'}{F} = \frac{G''}{G} = k.
\]
Taking into account the boundary condition, we want to find \( G(x) \) such that \( G'' = kG \) and \( G(0) = G(1) = 0 \). We know that it is possible only when \( k < 0 \) and \( k = -p^2 \). Then \( G(x) = \sin px \) and the condition \( G(1) = 0 \) implies \( p = \pi n \) for a positive integer \( n \).
Then \( F' = -p^2 F \) and \( F(t) = Ce^{-(\pi n)^2t} \). All solutions with divided variables are of the form
\[
u_n(x, t) = C_n \sin(\pi nx) e^{-(\pi n)^2t}.
\]

(b) Use the principle of superposition to find the solution of the boundary value problem that also satisfies the following initial condition
\[
u(x, 0) = 3 \sin(\pi x) + 5 \sin(4\pi x), \quad 0 < x < 1.
\]

\textit{Løsning} We know that a linear combination of solutions is a solution (superposition principle), to satisfy the the initial condition we take
\[
u(x, t) = 3 \sin(\pi x) e^{-\pi^2 t} + 5 \sin(4\pi x) e^{-(4\pi)^2 t}.
\]

Problem 3  Calculate the Fourier transform of the function
\[
f(x) = \begin{cases} e^{-2018x}, & x \geq 0 \\ 0, & x < 0. \end{cases}
\]

\textit{Løsning} By the definition
\[
\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-2018x - iwx} \, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{2018 + iw} = \frac{1}{\sqrt{2\pi}} \frac{2018 - iw}{2018^2 + w^2}.
\]
Problem 4  Assume that $f(z)$ is an analytic function in a domain $\Omega$ that satisfies $|f(z)| = 1$ for all $z \in \Omega$. Prove that $f$ is a constant.

Løsning  Consider the function $h(z) = \ln f(z)$, since $f$ has no zeros in $\Omega$ $h$ is defined and analytic in $\Omega$. Let $h(z) = u(z) + iv(z)$, where $u$ and $v$ are the real and imaginary parts of $h$. We have

\[ e^{u(z)}e^{iv(z)} = f(z). \]

Taking the absolute values we get $e^{u(z)} = |f(z)| = 1$. It implies that $u(z) = 0$ for all $z \in \Omega$. Then from the Cauchy-Riemann equation we see that $v_x = v_y = 0$ and $v$ is also a constant. Finally we conclude that $f(z) = e^{h(z)}$ is a constant.

Problem 5  Determine the radius of convergence of the Taylor series of the function

\[ h(z) = \frac{2}{1 + \cosh(z)} \]

centered at the origin. Explain your answer.

Løsning  $h(z)$ is the ratio of two analytic functions $f(z) = 2$ and $g(z) = 1 + \cosh z$. The second function, $g(z)$ is not zero at the origin, so $h$ is analytic at the origin, the Taylor series of $h$ at the origin converges in the largest disk centered at the origin where $h$ is analytic. To determine the radius of this disk we should calculate the distance from the origin to the closest singular point of $h$. The singular points of $h$ are the zeros of $g(z)$. We solve

\[ 1 + \cosh(z) = 0 \iff e^{z} = -1. \]

Let $z = x + iy$ then $e^{z} = e^{x}(\cos y + i \sin y)$, if $e^{z} = -1$ then $x = 0$ and $y = \pi + 2\pi k$ where $k$ is integer. The closest zeros of $g$ to the origin are points $i\pi$ and $-i\pi$. Thus the radius of convergence of the Taylor series centered at the origin is $\pi$.

Problem 6  Consider the function

\[ f(z) = \frac{\ln(z)}{z^2 + a^2}, \quad a > 0, \]

in the upper half plane. For $z = re^{i\theta}$, $0 \leq \theta \leq \pi$ we define

\[ \ln(z) = \ln(r) + i\theta. \]

Let $C$ be the following contour:
a) Determine $\oint_C f(z) \, dz$ by residue calculations.

*Løsning* First we find the singular points of $f$ in the upper half plane. There is only one, it is $z_0 = ia$. If $R < a$ and the contour does not contain the singular point the the integral is zero. If $R > a$ then by the residue theorem

$$\oint_C f(z) \, dz = 2\pi i \text{Res}_{ia} f(z).$$

Since $f(z)$ has a simple pole at $ia$, we have

$$\text{Res}_{ia} f(z) = \frac{\ln(ia)}{ia + ia} = \frac{\ln a + i\pi/2}{2ia}.$$

We obtain

$$\oint_C f(z) \, dz = \frac{\pi(2\ln a + i\pi)}{2a}.$$

b) Show that the integrals over the half circles $C_R$ and $C_\varepsilon$ approach zero as $R \to \infty$ and $\varepsilon \to 0$.

*Løsning* First consider the integral over $C_R$. We have for $|z| = R$

$$\left| \frac{\ln(z)}{z^2 + a^2} \right| \leq \frac{\ln(R) + \pi}{R^2 - a^2}.$$

Further, integrating over the half-circle of radius $R$ we get

$$\left| \int_{C_r} f(z) \, dz \right| \leq \frac{\pi R (\ln(R) + \pi)}{R^2 - a^2} \to 0 \quad R \to \infty.$$

For small $\varepsilon$ we see that when $|z| = \varepsilon$

$$\left| \frac{\ln(z)}{z^2 + a^2} \right| \leq \frac{|\ln(\varepsilon)| + \pi}{a^2 - \varepsilon^2}$$

and the integral is bounded by

$$\frac{\pi\varepsilon (|\ln \varepsilon| + \pi)}{(a^2 - \varepsilon^2)} \to \varepsilon \to 0.$$
c) Compute
\[ \int_0^\infty \frac{\ln(x) \, dx}{x^2 + a^2}. \]

Løsning We have
\[ \int_{-\infty}^0 \frac{\ln x \, dx}{x^2 + a^2} + \int_0^\infty \frac{\ln x \, dx}{x^2 + a^2} = \lim_{R \to \infty, \varepsilon \to 0} \oint_C f(z) \, dz = \frac{\pi(2 \ln a + i\pi)}{2a}. \]

On the other hand, for \( x < 0 \) we have \( \ln x = \ln |x| + i\pi \), thus
\[ \int_0^\infty \frac{\ln x}{x^2 + a^2} = \frac{1}{2} \text{Re} \left( \frac{\pi(2 \ln a + i\pi)}{2a} \right) = \frac{\pi \ln a}{2a}. \]