It is a convention that for real positive \( z = x \) the expression \( e^{\ln x} \) means \( e^{\ln x} \) where \( \ln x \) is the elementary real natural logarithm (that is, the principal value \( \ln z \) \((z = x > 0)\) in the sense of our definition). Also, if \( z = x \), the base of the natural logarithm, \( z = e^{\ln z} \) is conventionally regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number \( a \),

\[
\alpha^2 = e^{2 \ln \alpha}.
\]

We have now introduced the complex functions needed in practical work, some of them (\( e^z \), \( \cos z \), \( \sin z \), \( \cosh z \), \( \sinh z \)) entire (Sec. 13.5), some of them (tan \( z \), cot \( z \), tanh \( z \), coth \( z \)) analytic except at certain points, and one of them (ln \( z \)) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the inverse trigonometric and hyperbolic functions see the problem set.

### PROBLEM SET 13.7

#### VERIFICATIONS IN THE TEXT

1. Verify the computations in Example 1.
2. Verify (5) for \( z_1 = -i \) and \( z_2 = 1 \).
3. Prove analyticity of \( \ln z \) by means of the Cauchy–Riemann equations in polar form (Sec. 13.4).
4. Prove (4a) and (4b).

#### COMPLEX NATURAL LOGARITHMIC IN \( z \)

5–11. Principal Value \( \ln z \). Find \( \ln z \) when \( z \) equals

- 5. \( 7 \)
- 6. \( 8 + 8i \)
- 7. \( 8 - 8i \)
- 8. \( 1 + i \)
- 9. \( 0.6 + 0.8i \)
- 10. \( -15 \pm 0.1i \)
- 11. \( -e^z \)

12–16. All Values of \( \ln z \). Find all values and graph some of them in the complex plane.

12. \( \ln e \)
13. \( \ln 1 \)
14. \( \ln (-5) \)
15. \( \ln (5e^2) \)
16. \( \ln (4 - 3i) \)
17. Show that the set of values of \( \ln (5^2) \) differs from the set of values of \( 2 \ln z \).

18–21. Equations. Solve for \( z \).

18. \( \ln z = \frac{w}{2} \)
19. \( \ln z = 4 - 3i \)
20. \( \ln z = e + \pi i \)
21. \( \ln z = 0.4 + 0.2i \)


22. \( (2i)^{20} \)
23. \( (1 + i)^{1-i} \)
24. \( (1 - i)^{1+i} \)
25. \( (3-i)^{3+i} \)

For arithmetic operations with complex numbers

\[
z = x + iy = e^{\theta i} = r \cos \theta + i \sin \theta,
\]

\[
r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \arctan \left(\frac{y}{x}\right), \text{ and for their representation in the complex plane, see Secs. 13.1 and 13.2.}
\]

A complex function \( f(z) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \) if it has a derivative (Sec. 13.3).

\[
\frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \text{everywhere in } D.
\]

A point \( z = z_0 \) if it has a derivative in a neighborhood of \( z_0 \) (not merely at \( z_0 \) itself).
Bounds for Integrals. **ML-Inequality**

There will be a frequent need for estimating the absolute value of complex line integrals. The basic formula is

\[ \left| \int_C f(z) \, dz \right| \leq ML \]  

(ML-inequality)

**Proof**

Taking the absolute value in (2) and applying the generalized inequality (6*) in Sec. 14.1, we obtain

\[ |S| = \sum_{m=1}^{n} |f(z_m)| \Delta z_m \leq \sum_{m=1}^{n} |f(z_m)| |\Delta z_m| \leq M \sum_{m=1}^{n} |\Delta z_m| \]

Now \( |\Delta z_m| \) is the length of the chord whose endpoints are \( z_{m-1} \) and \( z_m \) (see Fig. 340). Hence the sum on the right represents the length \( L^* \) of the broken line of chords whose endpoints are \( z_0, z_1, \ldots, z_n \) (\(-2\)). If \( n \) approaches infinity in such a way that the greatest \( |\Delta z_m| \) and thus \( |\Delta z_m| \) approach zero, then \( L^* \) approaches the length \( L \) of the curve \( C \) by the definition of the length of a curve. From this the inequality (13) follows.

We cannot see from (13) how close to the bound \( ML \) the actual absolute value of the integral is, but this will be no handicap in applying (13). For the time being we explain the practical use of (13) by a simple example.

**Example 8**

**Estimation of an Integral**

Find an upper bound for the absolute value of the integral

\[ \int_C z^2 \, dz \]

the straight-line segment from \( 0 \) to \( 1 + i \), Fig. 344.

**Solution.** \( L = \sqrt{2} \) and \( |f(z)| = |z|^2 \geq 2 \) on \( C \) gives by (13)

\[ \left| \int_C z^2 \, dz \right| \leq 2 \sqrt{2} = 2.8284. \]

The absolute value of the integral is \( |\frac{-3}{2} + \frac{3}{2}i| = \sqrt{\frac{9}{4} + \frac{9}{4}} = \frac{3}{2} \sqrt{2} = 2.1213 \) (see Example 1).

**Summary on Integration.** Line integrals of \( f(z) \) can always be evaluated by (10), using a representation (1) of the path of integration. If \( f(z) \) is analytic, indefinite integration by (9) as in calculus will be simpler (proof in the next section).
Also, if \( G(z) = f(z) \), then \( F'(z) - G'(z) = 0 \) in \( D \); hence \( F(z) - G(z) \) is constant in \( D \), since Team Project 30 in Problem Set 13.4. That is, two indefinite integrals of \( f(z) \) can differ only by a constant. The latter drops out in (9) of Sec. 14.1, so that we can use any indefinite integral of \( f(z) \). This proves Theorem 3.

**Cauchy’s Integral Theorem for Multiply Connected Domains**

Cauchy’s theorem applies to multiply connected domains. We first explain this for a **doubly connected domain** \( D \) with outer boundary curve \( C_1 \) and inner curve \( C_2 \) (Fig. 353). If a function \( f(z) \) is analytic in any domain \( D^* \) that contains \( D \) and its boundary curves, we claim that

\[
\oint_{C_1} f(z) \, dz = \oint_{C_2} f(z) \, dz
\]

both integrals being taken counterclockwise (or both clockwise, and regardless of whether or not the full interior of \( C_2 \) belongs to \( D^* \)).

**Proof**

By two cuts \( C_1 \) and \( C_2 \) (Fig. 354) we cut \( D \) into two simply connected domains \( D_1 \) and \( D_2 \) in which and on whose boundaries \( f(z) \) is analytic. By Cauchy’s integral theorem, the integral over the entire boundary of \( D_1 \), taken in the sense of the arrows in Fig. 354, is zero, and so is the integral over the boundary of \( D_2 \), and thus their sum. In this sum the integrals over the cuts \( C_1 \) and \( C_2 \) cancel because we integrate over them in both directions—this is the key—and we are left with the integrals over \( C_1 \) (counterclockwise) and \( C_2 \) (clockwise; see Fig. 354); hence by reversing the integration over \( C_2 \) (to counterclockwise) we have

\[
\oint_{C_1} f(z) \, dz - \oint_{C_2} f(z) \, dz = 0
\]

and (6) follows.

For domains of higher connectivity the idea remains the same. Thus, for a **triplly connected domain** we use three cuts \( C_1, C_2, C_3 \) (Fig. 355). Adding integrals as before, the integrals over the cuts cancel and the sum of the integrals over \( C_1 \) (counterclockwise) and \( C_2, C_3 \) (clockwise) is zero. Hence the integral over \( C_1 \) equals the sum of the integrals over \( C_2 \) and \( C_3 \), all three now taken counterclockwise. Similarly for quadruply connected domains, and so on.

**POBLEM SET 14.2**

**COMMENTS ON TEXT AND EXAMPLES**

1. **Cauchy’s Integral Theorem.** Verify Theorem 1 for the integral of \( z^2 \) over the boundary of the square with vertices \( z = \pm 1 \). **Hint:** Use deformation.

2. For what contours \( C \) will it follow from Theorem 1 that

\[
\begin{align*}
(a) & \quad \oint_C \frac{dz}{z - 1} = 0, \\
(b) & \quad \oint_C \frac{dz}{z^2 + 4} = 0
\end{align*}
\]

3. **Deformation principle.** Can we conclude from Example 4 that the integral is also zero over the contour in Prob. 1?

4. If the integral of a function over the unit circle equals 2 and over the circle of radius 3 equals 6, can the function be analytic everywhere in the annulus \( 1 < |z| < 3 \)?

5. **Connectedness.** What is the connectedness of the domain in which \( \cos(z^2)/(z^2 + 1) \) is analytic?

6. **Path independence.** Verify Theorem 2 for the integral of \( \cos(z^2) \) from 0 to \( 1 + i \) (a) along the shortest path and (b) along the x-axis to 1 and then straight up to \( 1 + i \).

7. **Deformation.** Can we conclude in Example 2 that the integral of \( 1/(z^2 + 4) \) over (a) \( |z| = 2 \) and (b) \( |z| = 3 \) is zero?

8. **TEAM EXPERIMENT: Cauchy’s Integral Theorem.**

(a) **Main Aspects.** Each of the problems in Examples 1–3 explains a basic fact in connection with Cauchy’s theorem. Find five examples of your own, more complicated ones if possible, each illustrating one of these facts.

(b) **Partial fractions.** Write \( f(z) \) in terms of partial fractions and integrate it counterclockwise over the unit circle, where

\[
\begin{align*}
(f) & \quad f(z) = \frac{z + 2}{z^2 + 4}, \\
(g) & \quad f(z) = \frac{z + 1}{z^2 + 2z + 5}
\end{align*}
\]

9. **Deformation of path.** Review (c) and (d) of Team Project 34, Sec. 14.1, in the light of the principle of deformation of path. Then consider another family of paths with common endpoints, say, \( z = 1 + i(a - t^2), \ 0 \leq t \leq 1, a \) a real constant, and experiment with the integration of analytic and nonanalytic functions of your choice over these paths (e.g., \( z, \Im z, z^2, \Re z, \Im z^2 \), etc.).

**CAUCHY’S THEOREM APPLICABLE?**

Integrate \( f(z) \) counterclockwise around the unit circle. Indicate whether Cauchy’s integral theorem applies. Show the details.

9. \( f(z) = \exp(z^2) \)

10. \( f(z) = \tan(z^2) \)

11. \( f(z) = 1/(4z - 1) \)

12. \( f(z) = z^3 \)

13. \( f(z) = 1/(z^2 - 1) \)

14. \( f(z) = 1/z \)

15. \( f(z) = z \)

16. \( f(z) = 1/(z + 1) \)

17. \( f(z) = 1/z^2 \)

18. \( f(z) = 1/(z^2 - 1) \)

19. \( f(z) = z^2 \cot(z) \)

**FURTHER CONTOUR INTEGRALS**

Evaluate the integral. Does Cauchy’s theorem apply? Show details.

20. \( \int_{C} \frac{\cos(z)}{z - 1} \, dz \)

21. \( \int_{C} \frac{dz}{z - 2i} \)

22. \( \int_{C} \frac{z}{z - 2} \, dz \)

23. \( \int_{C} \frac{2z - 1}{z^2 + 4} \, dz \)
24. \[\int_C \frac{dz}{z^2 - 1}, \quad C:\]

Use partial fractions.

25. \[\int_C \frac{e^z}{z} \, dz, \quad C:\text{consists of } |z| = 1 \text{ counter-clockwise and } |z| = 1 \text{ clockwise.}\]

26. \[\int_C \cosh \frac{1}{2}z \, dz, \quad C:\text{the circle } |z - \frac{1}{2}pi| = 1 \text{ clockwise.}\]

14.3 Cauchy's Integral Formula

Cauchy's integral theorem leads to Cauchy's integral formula. This formula is useful for evaluating integrals as shown in this section. It has other important roles, such as in proving the surprising fact that analytic functions have derivatives of all orders, as shown in the next section, and in showing that all analytic functions have a Taylor series representation (to be seen in Sec. 15.4).

**Theorem 1**

**Cauchy's Integral Formula**

Let \( f(z) \) be analytic in a simply connected domain \( D \). Then for any point \( z_0 \) in \( D \) and any simple closed path \( C \) in \( D \) that encloses \( z_0 \) (Fig. 356),

\[
\int_C \frac{f(z)}{z-z_0} \, dz = 2\pi i f(z_0)
\]

(Cauchy’s integral formula)

the integration being taken counter-clockwise. Alternatively (for representing \( f(z_0) \) by a contour integral, divide (1) by \( 2\pi i \)),

\[
f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \, dz
\]

(Cauchy’s integral formula).

**Proof**

By addition and subtraction, \( f(z) = f(z_0) + \{f(z) - f(z_0)\} \). Inserting this into (1) on the left and taking the constant factor \( f(z_0) \) out from under the integral sign, we have

\[
\int_C \frac{f(z)}{z-z_0} \, dz = f(z_0) \int_C \frac{dz}{z-z_0} + \int_C \frac{f(z) - f(z_0)}{z-z_0} \, dz.
\]

The first term on the right equals \( f(z_0) \cdot 2\pi i \), which follows from Example 6 in Sec. 14.2 with \( m = 1 \). If we can show that the second integral on the right is zero, then it would prove the theorem. Indeed, we can. The integrand of the second integral is analytic, except

at \( z_0 \). Hence, by (6) in Sec. 14.2, we can replace \( C \) by a small circle \( K \) of radius \( \rho \) and center \( z_0 \) (Fig. 357), without altering the value of the integral. Since \( f(z) \) is analytic, it is continuous (Team Project 24, Sec. 13.3). Hence, an \( \epsilon > 0 \) being given, we can find a \( \delta > 0 \) such that \( |f(z) - f(z_0)| < \epsilon \) for all \( z \) in the disk \( |z - z_0| < \delta \). Choosing the radius \( \rho \) of \( K \) smaller than \( \delta \), we thus have the inequality

\[
\frac{|f(z) - f(z_0)|}{|z - z_0|} < \frac{\epsilon}{\rho}
\]

at each point of \( K \). The length of \( K \) is \( 2\pi \rho \). Hence, by the ML-inequality in Sec. 14.1,

\[
\left| \int_K \frac{f(z)}{z-z_0} \, dz \right| < \frac{\epsilon}{\rho} 2\pi \rho = 2\pi \epsilon.
\]

Since \( \epsilon > 0 \) can be chosen arbitrarily small, it follows that the last integral in (2) must have the value zero, and the theorem is proved.

**Example 1**

**Cauchy's Integral Formula**

\[
\int_C \frac{e^z}{z-2} \, dz = 2\pi i e^2 = 46.268i
\]

for any contour enclosing \( z_0 = 2 \) (since \( e^z \) is entire), and zero for any contour for which \( z_0 = 2 \) lies outside (by Cauchy's integral theorem).

**Example 2**

**Cauchy's Integral Formula**

\[
\int_C \frac{z^2 - 6}{z^2 - 1} \, dz = \frac{3\pi^2 - 3}{2} i
\]

\[
= \frac{3\pi^2 - 3}{2} i
\]

\[
= \pi - 6\pi i
\]

(C(\theta) = \frac{1}{2} i \text{ inside } C).

**Example 3**

**Integration Around Different Contours**

Integrate

\[
f(z) = \frac{z^2 + 1}{z^2 - 1} = \frac{z^2 + 1}{(z + 1)(z - 1)}
\]

clockwise around each of the four circles in Fig. 358.
EXAMPLE 5 Principle of Inverse Mapping. Mapping \( w = \ln z \)

**Principle.** The mapping by the inverse \( z = f^{-1}(w) \) of \( w = f(z) \) is obtained by interchanging the roles of the \( z \)-plane and the \( w \)-plane in the mapping by \( w = f(z) \).

Now the principal value \( w = f(z) = \ln z \) of the natural logarithm has the inverse \( z = f^{-1}(w) = e^w \). From Example 4 (with the notations \( z \) and \( w \) interchanged!) we know that \( f^{-1}(w) = e^w \) maps the fundamental region of the \( z \)-plane without \( z = 0 \) (because \( e^w \neq 0 \) for every \( w \)). Hence \( w = f(z) = \ln z \) maps the \( z \)-plane without the origin and cut along the negative real axis (where \( \theta = \ln \left| z \right| \) jumps by \( 2\pi \)) conformally onto the horizontal strip \( -\pi < \arg z < \pi \) of the \( w \)-plane, where \( w = u + iv \).

Since the mapping \( w = \ln z + 2\pi i \) differs from \( w = \ln z \) by the translation \( 2\pi i \) (vertically upward), this function maps the \( z \)-plane (cut as before and \( 0 \) omitted) onto the strip \( -\pi < \arg z < \pi \). Similarly for each of the infinitely many mappings \( w = \ln z + 2\pi i n \) (an \( n = 0, 1, 2, \cdots \)). The corresponding horizontal strips of width \( 2\pi \) (images of the \( z \)-plane under these mappings) cover the whole \( w \)-plane without overlapping.

**Magnification Ratio.** By the definition of the derivative we have

\[
\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).
\]

Therefore, the mapping \( w = f(z) \) magnifies (or shortens) the lengths of short lines by approximately the factor \( f'(z_0) \). The image of a small figure **conforms to** the original figure in the sense that it has approximately the same shape. However, since \( f'(z) \) varies from point to point, a large figure may have an image whose shape is not that of the figure from that point of view.

**More on the Condition \( f'(z) \neq 0 \).** From (4) in Sec. 13.4 and the Cauchy–Riemann equations we obtain

\[
|f'(z)|^2 = \left| \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right|^2 = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial u}{\partial x}
\]

that is,

\[
|f'(z)|^2 = \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial y} \right|^2.
\]

This determinant is the so-called **Jacobian** (Sec. 10.3) of the transformation \( w = f(z) \) written in real form \( u(x, y), v(x, y) \). Hence \( f'(z_0) \neq 0 \) implies that the Jacobian is not 0 at \( z_0 \). This condition is sufficient that the mapping \( w = f(z) \) in a sufficiently small neighborhood of \( z_0 \) is one-to-one or injective (different points have different images). See Ref. [GenReff4] in App. 1.

1. On Fig. 378. One "rectangle" and its image are colored. Identify the images for the other "rectangles."
2. **Mapping** \( w = z^2 \). Draw an analog of Fig. 378 for \( w = z^2 \).
3. **Conformality.** Why do the images of the straight lines \( x = \text{const} \) and \( y = \text{const} \) under a mapping by an analytic function intersect at right angles? Same question for the curves \( |z| = \text{const} \) and \( \arg z = \text{const} \). Are there exceptional points?
4. **Experiment on** \( w = z \). Find out whether \( w = z \) preserves angles in size as well as in sense. Try to prove your result.

### 17.2 Linear Fractional Transformations (Mobius Transformations)

Conformal mappings can help in modeling and solving boundary value problems by first mapping regions conformally onto another. We shall explain this for standard regions (disks, half-planes, strips) in the next section. For this it is useful to know properties of special basic mappings. Accordingly, let us begin with the following very important case.

The next two sections discuss linear fractional transformations. The reason for our thorough study is that such transformations are useful in modeling and solving boundary value problems, as we shall see in Chapter 18. The task is to get a good grasp of which conformal mappings map certain regions conformally onto each other, such as, say mapping a disk onto a half-plane (Sec. 17.3) and so forth. Indeed, the first step in the modeling process of solving boundary value problems is to identify the correct conformal mapping that is related to the "geometry" of the boundary value problem.

The following class of conformal mappings is very important. **Linear fractional transformations** (or Mobius Transformations) are mappings

\[
w = \frac{az + b}{cz + d} \quad (ad - bc \neq 0)
\]

where \( a, b, c, d \) are complex or real numbers. Differentiation gives...