If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number $z\ (\neq 0)$, then the n values of $\sqrt[n]{z}$ in (15) are

$$w_1, \quad w_1\omega, \quad w_1\omega^2, \quad \cdots, \quad w_1\omega^{n-1} \tag{17}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

PROBLEM SET 13.2

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$

2. $-2 + 2i$

3. $2i, \quad -2i$

4. -4

5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$

6. $\dfrac{\sqrt{5} - 10i}{-\frac{1}{2}\sqrt{5} + 5i}$

7. $1 + \frac{1}{2}\pi i$

8. $\dfrac{7 + 4i}{3 - 2i}$

9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $1 - i$

10. $-5, \quad -5 - i, \quad -5 + i$

11. $\sqrt{3} \pm i$

12. $-\pi - \pi i$

13. $(1 - i)^{20}$

14. $-1 + 0.1i, \quad -1 - 0.1i$

15–18 CONVERSION TO $x + iy$

Graph in the complex plane and represent in the form $x + iy$:

15. $4\,(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2})$

16. $6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$

17. $\sqrt{8}\,(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$

18. $\sqrt{50}\,(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$

ROOTS

19. CAS PROJECT. Roots of Unity and Their Graphs. Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

20. TEAM PROJECT. Square Root. (a) Show that $w = \sqrt{z}$ has the values

$$\begin{aligned} w_1 &= \sqrt{r}\left[\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right], \\ w_2 &= \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i \sin\left(\frac{\theta}{2} + \pi\right)\right] \\ &= -w_1. \end{aligned} \tag{18}$$

(b) Obtain from (18) the often more practical formula

$$\sqrt{z} = \pm[\sqrt{\tfrac{1}{2}(|z| + x)} + (\operatorname{sign} y) i \sqrt{\tfrac{1}{2}(|z| + x)}] \tag{19}$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $-14i$, $-9 - 40i$, and $1 + \sqrt{48}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

21–27 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt[3]{1 - i}$

22. $\sqrt[3]{3 + 4i}$

23. $\sqrt[3]{343}$

24. $\sqrt[4]{-4}$

25. $\sqrt[4]{i}$

26. $\sqrt[8]{1}$

27. $\sqrt[5]{-1}$

28–31 EQUATIONS

Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$

29. $z^2 - z + 1 + i = 0$

30. $z^4 + 324 = 0$. Using the solutions, factor $z^4 + 324$ into quadratic factors with *real* coefficients.

31. $z^4 - 6iz^2 + 16 = 0$

32–35 INEQUALITIES AND EQUALITY

32. Triangle inequality. Verify (6) for $z_1 = 3 + i$, $z_2 = -2 + 4i$

33. Triangle inequality. Prove (6).

34. Re and Im. Prove $|\operatorname{Re} z| \leqq |z|$, $|\operatorname{Im} z| \leqq |z|$.

35. Parallelogram equality. Prove and explain the name

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2).$$

13.3 Derivative. Analytic Function

Just as the study of calculus or real analysis required concepts such as domain, neighborhood, function, limit, continuity, derivative, etc., so does the study of complex analysis. Since the functions live in the complex plane, the concepts are slightly more difficult or *different* from those in real analysis. This section can be seen as a reference section where many of the concepts needed for the rest of Part D are introduced.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 330) has already occurred in Sec. 13.2. Figure 331 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$

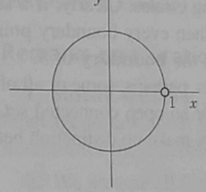

Fig. 330. Unit circle

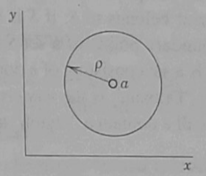

Fig. 331. Circle in the complex plane

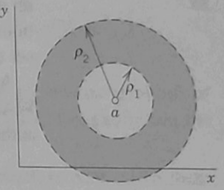

Fig. 332. Annulus in the complex plane

because it is the set of all z whose distance $|z - a|$ from the center a equals ρ. Accordingly, its interior (**"open circular disk"**) is given by $|z - a| < \rho$, its interior plus the circle itself (**"closed circular disk"**) by $|z - a| \leqq \rho$, and its exterior by $|z - a| > \rho$. As an example, sketch this for $a = 1 + i$ and $\rho = 2$, to make sure that you understand these inequalities.

An open circular disk $|z - a| < \rho$ is also called a **neighborhood** of a or, more precisely, a *ρ-neighborhood* of a. And a has infinitely many of them, one for each value of $\rho\ (> 0)$, and a is a point of each of them, by definition!

In modern literature *any set* containing a ρ-neighborhood of a is also called a *neighborhood* of a.

Figure 332 shows an **open annulus** (circular ring) $\rho_1 < |z - a| < \rho_2$, which we shall need later. This is the set of all z whose distance $|z - a|$ from a is greater than ρ_1 but less than ρ_2. Similarly, the **closed annulus** $\rho_1 \leqq |z - a| \leqq \rho_2$ includes the two circles.

Half-Planes. By the (open) *upper* **half-plane** we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the *lower half-plane*, $x > 0$ the *right half-plane*, and $x < 0$ the *left half-plane*.

To get rid of u_y, multiply (6a) by u and (6b) by v and add. Similarly, to eliminate u_x, multiply (6a) by $-v$ and (6b) by u and add. This yields

$$(u^2 + v^2)u_x = 0,$$
$$(u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$. If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$. Hence, by the Cauchy–Riemann equations, also $u_x = v_y = 0$. Together this implies $u = \text{const}$ and $v = \text{const}$; hence $f = \text{const}$. ■

We mention that, if we use the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$, then the **Cauchy–Riemann equations** are (Prob. 1)

$$(7)\qquad \begin{aligned} u_r &= \frac{1}{r}v_\theta, \\ v_r &= -\frac{1}{r}u_\theta \end{aligned} \qquad (r > 0).$$

Laplace's Equation. Harmonic Functions

The great importance of complex analysis in engineering mathematics results mainly from the fact that both the real part and the imaginary part of an analytic function satisfy Laplace's equation, the most important PDE of physics. It occurs in gravitation, electrostatics, fluid flow, heat conduction, and other applications (see Chaps. 12 and 18).

M 3 **Laplace's Equation**

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy **Laplace's equation**

$$(8)\qquad \nabla^2 u = u_{xx} + u_{yy} = 0$$

(∇^2 read "nabla squared") *and*

$$(9)\qquad \nabla^2 v = v_{xx} + v_{yy} = 0,$$

in D and have continuous second partial derivatives in D.

OOF Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y, we have

$$(10)\qquad u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy}.$$

Now the derivative of an analytic function is itself analytic, as we shall prove later (in Sec. 14.4). This implies that u and v have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal: $v_{yx} = v_{xy}$. By adding (10) we thus obtain (8). Similarly, (9) is obtained by differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x and subtracting, using $u_{xy} = u_{yx}$. ■

Solutions of Laplace's equation having *continuous* second-order partial derivatives are called **harmonic functions** and their theory is called **potential theory** (see also Sec. 12.11). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **harmonic conjugate function** of u in D. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

EXAMPLE 4 **How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations**

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$ ■

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens a wide range of engineering and physical applications, as shown in Chap. 18.

PROBLEM SET 13.4

1. **Cauchy–Riemann equations in polar form.** Derive (7) from (1).

2–11 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? Use (1) or (7).

2. $f(z) = iz\bar{z}$
3. $f(z) = e^{-x}\cos(y) - ie^{-x}\sin(y)$
4. $f(z) = e^{x}(\cos y - i\sin y)$
5. $f(z) = \text{Re}(z^2) - i\,\text{Im}(z^2)$
6. $f(z) = 1/(z - z^5)$
7. $f(z) = -i/z^4$
8. $f(z) = \text{Arg}\, z$
9. $f(z) = 3\pi^2/(z^3 + 4\pi^2 z)$
10. $f(z) = \ln|z| + i\,\text{Arg}\, z$
11. $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

12–19 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = x^3 + y^3$
13. $u = -2xy$
14. $v = xy$
15. $u = -\frac{x}{x^2 + y^2}$
16. $u = \sin x \cosh y$
17. $v = (2x - 1)y$
18. $u = x^3 - 3xy^2$
19. $v = e^{-x}\sin 2y$
20. **Laplace's equation.** Give the details of the derivative of (9).

21–24 Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{-\pi x}\cos ay$
22. $u = \cos ax \cosh 2y$
23. $u = ax^3 + bxy$
24. $u = \cosh ax \cos y$
25. **CAS PROJECT. Equipotential Lines.** Write a program for graphing equipotential lines $u = \text{const}$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to **(a)** $u = x^2 - y^2$, $v = 2xy$, **(b)** $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$.
26. Apply the program in Prob. 25 to $u = e^x\cos y$, $v = e^x\sin y$ and to an example of your own.

Periodicity of e^x with period $2\pi i$,

$$e^{z+2\pi i} = e^z \qquad \text{for all } z \tag{12}$$

is a basic property that follows from (1) and the periodicity of cos y and sin y. Hence all the values that $w = e^z$ can assume are already assumed in the horizontal strip of width 2π

$$-\pi < y \leqq \pi \qquad \text{(Fig. 336).} \tag{13}$$

This infinite strip is called a **fundamental region** of e^z.

APLE 1 Function Values. Solution of Equations

Computation of values from (1) provides no problem. For instance,

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.8253 - 0.5646i) = 3.347 - 2.289i$$

$$|e^{1.4-1.6i}| = e^{1.4} = 4.055, \qquad \operatorname{Arg} e^{1.4-0.6i} = -0.6.$$

To illustrate (3), take the product of

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \qquad \text{and} \qquad e^{4-i} = e^4(\cos 1 - i \sin 1)$$

and verify that it equals $e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$.

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5$, $x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x \cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors. ■

To summarize: many properties of $e^z = \exp z$ parallel those of e^x; an exception is the periodicity of e^z with $2\pi i$, which suggested the concept of a fundamental region. Keep in mind that e^z is an *entire function.* (Do you still remember what that means?)

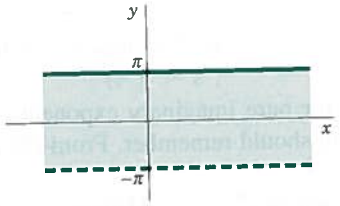

Fig. 336. Fundamental region of the exponential function e^z in the z-plane

OBLEM SET 13.5

s **entire.** Prove this.

Function Values. Find e^z in the form $u + iv$ if z equals

$4i$

$- 1.8i$

$\pi i/2$

3. $2\pi i(1 - i)$

5. $1 - 3\pi i$

7. $\sqrt{3} - \frac{\pi}{2}i$

8–13 **Polar Form.** Write in exponential form (6):

8. $\sqrt[n]{z}$

9. $3 - 4i$

10. $\sqrt{i}$, $\sqrt{-i}$

11. $-\frac{3}{2}$

12. $1/(1 - z)$

13. $1 - i$

14–17 **Real and Imaginary Parts.** Find Re and Im of

14. $e^{-\pi z}$

15. $\exp(-z^2)$

16. $e^{1/z}$

17. $\exp(z^3)$

18. TEAM PROJECT. Further Properties of the Exponential Function. (a) Analyticity. Show that e^z is entire. What about $e^{1/z}$? $e^{\bar{z}}$? $e^x(\cos ky + i \sin ky)$? (Use the Cauchy–Riemann equations.)

(b) Special values. Find all z such that (i) e^z is real, (ii) $|e^{-z}| < 1$, (iii) $e^{\bar{z}} = \overline{e^z}$.

(c) Harmonic function. Show that $u = e^{xy} \cos(x^2/2 - y^2/2)$ is harmonic and find a conjugate.

(d) Uniqueness. It is interesting that $f(z) = e^z$ is uniquely determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this using the Cauchy–Riemann equations.

19–22 **Equations.** Find all solutions and graph some of them in the complex plane.

19. $e^z = 1$

20. $e^z = 4 + 3i$

21. $e^z = 0$

22. $e^z = -2$

13.6 Trigonometric and Hyperbolic Functions. Euler's Formula

Just as we extended the real e^x to the complex e^z in Sec. 13.5, we now want to extend the familiar *real* trigonometric functions to *complex trigonometric functions.* We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}). \tag{1}$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex.

Furthermore, as in calculus we define

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z} \tag{2}$$

and

$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}. \tag{3}$$

Since e^z is entire, cos z and sin z are entire functions. tan z and sec z are not entire; they are analytic except at the points where cos z is zero; and cot z and csc z are analytic except

ROBLEM SET 13.6

4 **FORMULAS FOR HYPERBOLIC FUNCTIONS**

w that

$$\cosh z = \cosh x \cos y + i \sinh x \sin y$$

$$\sinh z = \sinh x \cos y + i \cosh x \sin y.$$

$$\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

$\cosh^2 z - \sinh^2 z = 1, \quad \cosh^2 z + \sinh^2 z = \cosh 2z$

Entire Functions. Prove that $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ are entire.

Harmonic Functions. Verify by differentiation that $\operatorname{Im} \cos z$ and $\operatorname{Re} \sin z$ are harmonic.

-12 **Function Values.** Find, in the form $u + iv$,

. $\sin \frac{\pi}{2} i$

7. $\cos (-i), \quad \sin (-i)$

. $\cos \pi i, \quad \cosh \pi i$

. $\cosh (-2 + i), \quad \cos (-1 - 2i)$

. $\sinh (3 + 4i), \quad \cosh (3 + 4i)$

11. $\sin \frac{\pi}{4} i, \quad \cos \left(\frac{\pi}{2} - \frac{\pi}{4} i\right)$

12. $\cos \frac{1}{2} \pi i, \quad \cos \left[\frac{1}{2} \pi (1 + i)\right]$

13–15 **Equations and Inequalities.** Using the definitions, prove:

13. $\cos z$ is even, $\cos (-z) = \cos z$, and $\sin z$ is odd, $\sin (-z) = -\sin z$.

14. $|\sinh y| \leqq |\cos z| \leqq \cosh y, |\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.

15. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin (z_1 + z_2) + \sin (z_1 - z_2)]$

16–19 **Equations.** Find all solutions.

16. $\sin z = 100$

17. $\cosh 2z = 0$

18. $\cosh z = -1$

19. $\sinh z = 0$

20. **Re tan *z* and Im tan *z*.** Show that

$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$

$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

3.7 Logarithm. General Power. Principal Value

We finally introduce the *complex logarithm*, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; see Sec. 13.5.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now, from Sec. 13.5, we know that e^{u+iv} has the absolute value e^u and the argument v. These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

$e^u = r$ gives $u = \ln r$, where $\ln r$ is the familiar *real* natural logarithm of the positive number $r = |z|$. Hence $w = u + iv = \ln z$ is given by

$$\ln z = \ln r + i\theta \qquad (r = |z| > 0, \quad \theta = \arg z). \tag{1}$$

Now comes an important point (without analog in real calculus). Since the argument of z is determined only up to integer multiples of 2π, ***the complex natural logarithm*** $\ln z$ ($z \neq 0$) ***is infinitely many-valued.***

The value of $\ln z$ corresponding to the principal value $\operatorname{Arg} z$ (see Sec. 13.2) is denoted by $\operatorname{Ln} z$ (Ln with capital L) and is called the **principal value** of $\ln z$. Thus

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z \qquad (z \neq 0). \tag{2}$$

The uniqueness of $\operatorname{Arg} z$ for given z ($\neq 0$) implies that $\operatorname{Ln} z$ is single-valued, that is, a function in the usual sense. Since the other values of $\arg z$ differ by integer multiples of 2π, the other values of $\ln z$ are given by

$$\ln z = \operatorname{Ln} z \pm 2n\pi i \qquad (n = 1, 2, \cdots). \tag{3}$$

They all have the same real part, and their imaginary parts differ by integer multiples of 2π.

If z is positive real, then $\operatorname{Arg} z = 0$, and $\operatorname{Ln} z$ becomes identical with the real natural logarithm known from calculus. If z is negative real (so that the natural logarithm of calculus is not defined!), then $\operatorname{Arg} z = \pi$ and

$$\operatorname{Ln} z = \ln |z| + \pi i \qquad (z \text{ negative real}).$$

From (1) and $e^{\ln r} = r$ for positive real r we obtain

$$e^{\ln z} = z \tag{4a}$$

as expected, but since $\arg (e^z) = y \pm 2n\pi$ is multivalued, so is

$$\ln (e^z) = z \pm 2n\pi i, \qquad n = 0, 1, \cdots. \tag{4b}$$

EXAMPLE 1 **Natural Logarithm. Principal Value**

$$\begin{aligned}
\ln 1 &= 0, \pm 2\pi i, \pm 4\pi i, \cdots & \operatorname{Ln} 1 &= 0 \\
\ln 4 &= 1.386294 \pm 2n\pi i & \operatorname{Ln} 4 &= 1.386294 \\
\ln (-1) &= \pm \pi i, \pm 3\pi i, \pm 5\pi i, \cdots & \operatorname{Ln} (-1) &= \pi i \\
\ln (-4) &= 1.386294 \pm (2n + 1)\pi i & \operatorname{Ln} (-4) &= 1.386294 + \pi i \\
\ln i &= \pi i/2, -3\pi/2, 5\pi i/2, \cdots & \operatorname{Ln} i &= \pi i/2 \\
\ln 4i &= 1.386294 + \pi i/2 \pm 2n\pi i & \operatorname{Ln} 4i &= 1.386294 + \pi i/2 \\
\ln (-4i) &= 1.386294 - \pi i/2 \pm 2n\pi i & \operatorname{Ln} (-4i) &= 1.386294 - \pi i/2 \\
\ln (3 - 4i) &= \ln 5 + i \arg (3 - 4i) & \operatorname{Ln} (3 - 4i) &= 1.609438 - 0.927295i \\
&= 1.609438 - 0.927295i \pm 2n\pi i & &\text{(Fig. 337)}
\end{aligned}$$

■