13.3 Derivative. Analytic Function

Inequalities and Equality
32. Triangle inequality. Verify (6) for $z_1 = 3 + i$.
$$z_2 = -2 + 4i$$

33. Triangle inequality. Prove (6).
$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Circles and Disks. Half-Planes

The unit circle $|z| = 1$ (Fig. 330) has already occurred in Sec. 13.2. Figure 331 shows a general circle of radius $r$ and center $a$. Its equation is
$$|z - a| = r$$

because it is the set of all $z$ whose distance $|z - a|$ from the center $a$ equals $r$. Accordingly, its interior ("open circular disk") is given by $|z - a| < r$, its interior plus the circle itself ("closed circular disk") by $|z - a| \leq r$, and its exterior by $|z - a| > r$. As an example, sketch this for $a = 1 + i$ and $r = 2$, to make sure that you understand these

inequalities.

An open circular $|z - a| < r$ is also called a neighborhood of $a$, or, more precisely, a $r$-neighborhood of $a$. And a has infinitely many of them, one for each value of $r$ ($> 0$), and $a$ is a point of each of them, by definition!

In modern literature any set containing a $r$-neighborhood of $a$ is also called a neighborhood of $a$.

Figure 332 shows an open annulus (circular ring) $r_0 < |z - a| < r_2$, which we shall need later. This is the set of all $z$ whose distance $|z - a|$ from $a$ is greater than $r_0$ but less than $r_2$. Similarly, the closed annulus $r_1 \leq |z - a| \leq r_2$ includes the two circles.

Halff-Planes.
By the (open) upper half-plane we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the lower half-plane, $x > 0$ the right half-plane, and $x < 0$ the left half-plane.
Laplace’s Equation. Harmonic Functions

The great importance of complex analysis in engineering mathematics results mainly from the fact that both the real part and the imaginary part of an analytic function satisfy Laplace’s equation, the most important PDE of physics. It occurs in gravitation, electrostatics, fluid flow, heat conduction, and other applications (see Chaps. 12 and 18).

Laplace’s Equation

If \( f(z) = u(x, y) + iv(x, y) \) is analytic in a domain \( D \), then both \( u \) and \( v \) satisfy Laplace’s equation

\[
\nabla^2 u = u_{xx} + u_{yy} = 0
\]

(8)

(\( \nabla^2 \) read “nabla squared”) and

\[
\nabla^2 v = v_{xx} + v_{yy} = 0,
\]

(9)

in \( D \) and have continuous second partial derivatives in \( D \).

D OF

Differentiating \( u_x = v_y \) with respect to \( x \) and \( u_y = -v_x \) with respect to \( y \), we have

\[
\begin{align*}
   u_{xx} &= v_{xy}, \\
   u_{yy} &= -v_{yx},
\end{align*}
\]

(10)

Now the derivative of an analytic function is itself analytic, as we shall prove later (in Sec. 14.4). This implies that \( u \) and \( v \) have continuous partial derivatives of all orders. In particular, the mixed second derivatives are equal: \( v_{xy} = v_{yx} \). By adding (10) we thus obtain (8). Similarly, (9) is obtained by differentiating \( u_x = v_y \) with respect to \( x \) and \( u_y = -v_x \) with respect to \( y \) and subtracting, using \( v_{xy} = v_{yx} \).

Solutions of Laplace’s equation having continuous second-order partial derivatives are called harmonic functions and their theory is called potential theory (see also Sec. 12.11). Hence the real and imaginary parts of an analytic function are harmonic functions.

12-19

HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function \( f(z) \) where \( u(x, y) = f(x, y) \), \( v(x, y) = f(x, y) \), or \( u(x, y) + iv(x, y) \). Are the functions harmonic if \( u(x, y) = x^2 + y^2 \), \( v(x, y) = 2xy \), \( u(x, y) = x^3 - 3xy^2 \), \( v(x, y) = 3x^2y - y^3 \), \( u(x, y) = e^x \cos y \), \( v(x, y) = e^x \sin y \), and \( u(x, y) = x \sin \theta + y \cos \theta \)?

21-24

Determining a and \( b \) so that the given function is harmonic: and find a harmonic conjugate.

14. \( u = x^2 + y^2 \)
15. \( u = \frac{x^2}{x^2 + y^2} \)
16. \( u = \sin x \cos y \)
17. \( v = (2x - 1)y \)
18. \( u = x^3 - 3xy^2 \)
19. \( v = e^{-x^2} \sin 2y \)
20. Laplace’s equation. Give the details of the derivative of (9).

25. CAS PROJECT. Equipotential Lines. Write a program for graphing equipotential lines \( u = \text{const} \) of a harmonic function \( u \) and of its conjugate \( v \) on the same axes. Apply the program to (a) \( u = e^x \cos y \), \( v = e^x \sin y \), (b) \( u = x^2 - 3xy^2 \), \( v = 3x^2y - y^3 \).

26. Apply the program in Prob. 25 to \( u = e^x \cos y \), \( v = e^x \sin y \) and to an example of your own.
Periodicity of $e^z$ with period $2\pi i$,

$$e^{z+2\pi i} = e^z \quad \text{for all } z$$

is a basic property that follows from (1) and the periodicity of $\cos y$ and $\sin y$. Hence all the values of $w = e^z$ can assume are already assumed in the horizontal strip of width $2\pi$.

$$-\pi < y \leq \pi$$  \hspace{1cm} (Fig. 336).

This infinite strip is called a fundamental region of $e^z$.

**Function Values. Solution of Equations**

Computation of values from (1) provides no problem. For instance,

$$e^{1.4 + 0i} = e^{1.4} \cos(0.6) - i \sin(0.6) = 4.035(0.8253 - 0.5646) = 3.247 - 2.289i$$

$$|e^{1.4 + 0i}| = e^{1.4} = 4.055, \quad \arg e^{1.4 + 0i} = -0.6.$$

To illustrate (3), take the product of

$$e^{3+2i} = e^3 \cos(1 + i \sin 1)$$

and

$$e^{4-i} = e^4 \cos(1 - i \sin 1)$$

and verify that it equals $e^{7} = \cos(\frac{7\pi}{2} + \frac{\pi}{2}) = i$.

To solve the equation $e^z = 3 + 4i$, note first that $|3 + 4i| = 5$, $\arg(3 + 4i) = 53.13^\circ$, $\sin(53.13^\circ) = 0.8$, and $\cos(53.13^\circ) = 0.6$. The given equation can then be solved graphically by using the natural logarithm function.

**13.6 Trigonometric and Hyperbolic Functions. Euler’s Formula**

Just as we extended the real $e^z$ to the complex $e^z$ in Sec. 13.5, we now want to extend the familiar real trigonometric functions to complex trigonometric functions. We can do this by the use of the Euler formulas (Sec. 13.5)

$$e^{iz} = \cos x + i \sin x, \quad e^{-iz} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the real cosine and sine

$$\cos x = \frac{1}{2} (e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i} (e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i} (e^{iz} - e^{-iz}).$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex. Furthermore, as in calculus we define

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

and

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

Since $e^z$ is entire, $\cos z$ and $\sin z$ are entire functions, $\tan z$ and $\sec z$ are not entire; they are analytic except at the points where $\cos z$ is zero, and $\cot z$ and $\csc z$ are analytic except
3.7 Logarithm. General Power. Principal Value

We finally introduce the complex logarithm, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through the section with extra care).

The natural logarithm of \( z = x + iy \) is denoted by \( \ln z \) (sometimes also by \( \log z \)) and is defined as the inverse of the exponential function; that is, \( w = \ln z \) is defined for \( z \neq 0 \) by the relation

\[ e^w = z. \]

(Note that \( z = 0 \) is impossible, since \( e^w \neq 0 \) for all \( w \); see Sec. 13.5.) If we set \( w = u + iv \) and \( z = re^{i\theta} \), this becomes

\[ e^w = e^{u+iv} = re^{i\theta}. \]

Now, from Sec. 13.5, we know that \( e^{u+iv} \) has the absolute value \( e^u \) and the argument \( v \). These must be equal to the absolute value and argument of the right:

\[ e^u = r \quad \text{and} \quad v = \theta. \]