This shows that the expressions in the parentheses must be the Fourier coefficients $b_n$ of $f(x)$; that is, by (4) in Sec. 11.3,

$$b_n = A_n \sin \frac{n\pi b}{a} = \frac{2}{a} \int_{0}^{a} f(x) \sin \frac{n\pi x}{a} \, dx.$$ 

From this and (16) we see that the solution of our problem is

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b},$$

where

$$A_{nm} = \frac{2}{a} \int_{0}^{a} f(x) \sin \frac{n\pi x}{a} \, dx.$$ 

We have obtained this solution formally, neither considering convergence nor showing that the series for $u$, $u_{xx}$, and $u_{yy}$ have the right sums. This can be proved if one assumes that $f$ and $f'$ are continuous and $f''$ is piecewise continuous on the interval $0 \leq x \leq a$. The proof is somewhat involved and relies on uniform convergence. It can be found in [C4] listed in App. 1.

**Unifying Power of Methods. Electrostatics, Elasticity**

The Laplace equation (14) also governs the electrostatic potential of electrical charges in any region that is free of these charges. Thus our steady-state heat problem can also be interpreted as an electrostatic potential problem. Then (17), (18) is the potential in the rectangle R when the upper side of R is at potential $f(x)$ and the other three sides are grounded. Actually, in the steady-state case, the two-dimensional wave equation (to be considered in Secs. 12.8, 12.9) also reduces to (14). Then (17), (18) is the displacement of a rectangular elastic membrane (rubber sheet, drumhead) that is fixed along its boundary, with three sides lying in the $x$-$y$-plane and the fourth side given the displacement $f(x)$.

This is another impressive demonstration of the unifying power of mathematics. It illustrates that entirely different physical systems may have the same mathematical model and can thus be treated by the same mathematical methods.

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**PROBLEM SET 12.6**

1. Decay. How does the rate of decay of (8) with fixed $n$ depend on the specific heat, the density, and the thermal conductivity of the material?

2. Decay. If the first eigenfunction (8) of the bar decreases to half its value within 20 sec, what is the value of the diffusivity?

3. Eigenfunctions. Sketch or graph and compare the first three eigenfunctions (8) with $B_n = 1, c = 1$, and $L = \pi$ for $x = 0, 0.1, 0.2, \ldots, 1.0$.

4. WRITING PROJECT. Wave and Heat Equations. Compare these PDEs with respect to general behavior of eigenfunctions and kind of boundary and initial conditions. State the difference between Fig. 291 in Sec. 12.3 and Fig. 295.

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**SEC. 12.6 Heat Equation: Solution by Fourier Series**

18-25 **TWO-DIMENSIONAL PROBLEMS**

18. Laplace equation. Find the potential in the rectangle $0 \leq x \leq 20, 0 \leq y \leq 20$ if the upper side is kept at potential 110 V and whose other sides are grounded.

19. Find the potential in the square $0 \leq x \leq 20, 0 \leq y \leq 20$ if the upper side is kept at the potential 10000$\frac{V}{m}$ and the other sides are grounded.

20. CAS PROJECT. Isotherms. Find the steady-state solutions (temperatures) in the square plate in Fig. 297 with $a = 2$ satisfying the following boundary conditions. Graph isotherms.

(a) $u = 80 \sin \pi x$ on the upper side, 0 on the others.
(b) $u = 0$ on the vertical sides, assuming that the other sides are perfectly insulated.
(c) Boundary conditions of your choice (such that the solution is not identically zero).

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**Fig. 297. Square plate**

21. Heat flow in a plate. The faces of the thin square plate in Fig. 297 with side $a = 24$ are perfectly insulated. The upper side is kept at $25^\circ C$ and the other sides are kept at $0^\circ C$. Find the steady-state temperature $u(x, y)$ in the plate.

22. Find the steady-state temperature in the plate in Prob. 21 if the lower side is kept at $0^\circ C$, the upper side at $12^\circ C$, and the other sides are kept at $0^\circ C$. Hint: Split into two problems in which the boundary temperature is 0 or on three sides each.

23. Mixed boundary value problem. Find the steady-state temperature in the plate in Prob. 21 with the upper and lower sides perfectly insulated, the left side kept at $0^\circ C$, and the right side kept at $f(x^2/4^\circ C)$.

24. Radiation. Find steady-state temperatures in the rectangle in Fig. 296 with the upper and left sides perfectly insulated and the right side radiating into a medium at $0^\circ C$ according to $u_{y}(x, y) = 0$, $h > 0$ constant. (You will get many solutions since no condition on the lower side is given.)

25. Find formulas similar to (17), (18) for the temperature in the rectangle $R$ of the text when the lower side of $R$ is kept at temperature $f(x)$ and the other sides are kept at $0^\circ C$. 

If \( w \) denotes the value corresponding to \( k = 1 \) in (16), then the \( n \) values of \( \sqrt[n]{z} \) can be written as

\[
\begin{align*}
1, & w, w^2, \ldots, w^{n-1}, \\
& \text{More generally, if } w_1 \text{ is any } n^{th} \text{ root of an arbitrary complex number } z \neq 0, \text{ then the } n \\
& \text{values of } \sqrt[n]{z} \text{ in (15) are} \\
& w_1, \quad w_1^2, \quad w_1^3, \quad \ldots, \quad w_1^{n-1}, \\
& \text{because multiplying } w_1 \text{ by } w^k \text{ corresponds to increasing the argument of } w_1 \text{ by } 2k\pi/n. \quad \text{Formula (17) motivates the introduction of roots of unity and shows their usefulness.}
\end{align*}
\]

13.3 Derivative. Analytic Function

Just as the study of calculus or real analysis required concepts such as domain, neighborhood, function, limit, continuity, derivative, etc., so does the study of complex analysis. Since the functions live in the complex plane, the concepts are slightly more difficult or different from those in real analysis. This section can be seen as a reference section where many of the concepts needed for the rest of Part D are introduced.

Circles and Disks. Half-Planes

The unit circle \(|z| = 1\) (Fig. 330) has already occurred in Sec. 13.2. Figure 331 shows a general circle of radius \( r \) and center \( a \). Its equation is

\[ |z - a| = r \]

because it is the set of all \( z \) whose distance \(|z - a|\) from the center \( a \) equals \( r \). Accordingly, its interior ("open circular disk") is given by \(|z - a| < r\), its interior plus the circle itself ("closed circular disk") by \(|z - a| \leq r\), and its exterior by \(|z - a| > r\). As an example, sketch this for \( a = 1 + i \) and \( r = 2 \), to make sure that you understand these inequalities.

An open circular disk \(|z - a| < r\) is also called a neighborhood of \( a \) or, more precisely, a \( r \)-neighborhood of \( a \). And a is infinitely many of them, one for each value of \( r (> 0) \), and \( a \) is a point of each of them, by definition!

In modern literature any set containing a \( r \)-neighborhood of \( a \) is also called a neighborhood of \( a \).

Figure 332 shows an open annulus (circular ring) \( |z - a| < r_2 \), which we shall need later. This is the set of all \( z \) whose distance \(|z - a|\) from \( a \) is greater than \( r_1 \) but less than \( r_2 \). Similarly, the closed annulus \( |z - a| \leq r_2 \) includes the two circles.
EXAMPLE 5
Polynomials, Rational Functions

The nonnegative integer powers 1, z, z^2, ... are analytic in the entire complex plane, and so are polynomials,
that is, functions of the form

\[ f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n \]

where c_0, c_1, ..., c_n are complex constants.

The quotient of two polynomials g(z) and h(z),

\[ f(z) = \frac{g(z)}{h(z)} \]

called a rational function. This f is analytic except at the points where h(z) = 0; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters.

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1-8 REGIONS OF PRACTICAL INTEREST
Determine and sketch or graph the sets in the complex plane given by

1. \[ |z + 1 - 2i| \leq \frac{1}{2} \]
2. \[ 0 < |z| < 1 \]
3. \[ 3 < |z - z_0| < 5 \]
4. \[ -\pi < \arg z < \pi \]
5. \[ |2z + 1| \leq 1 \]
6. \[ |z| + |z| = 1 \]
7. \[ z = i \]
8. \[ |z + i| \geq |z - i| \]

9. WRITING PROJECT. Sets in the Complex Plane. Write an essay formulating the corresponding portions of the text in your own words and illustrating them with examples of your own.

COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10-12 Function Values. Find Re f, and Im f, and their values at the given point z.
10. \[ f(z) = z + 2iz - 1 \]
11. \[ f(z) = 1 + i \]
12. \[ f(z) = (z - 1)(z + 1) \]

13. CAS PROJECT. Graphing Functions. Find and graph Re f, Im f, and |f| as surfaces over the z-plane. Also graph the two families of curves Re f(z) = const and Im f(z) = const in the same figure, and the curves |f(z)| = const in another figure, where (a) f(z) = z^2,
(b) f(z) = 1/z,
(c) f(z) = z^3.

14-17 Continuity. Find out, and give reason, whether f(z) is continuous at z = 0 if f(z) = 0 and for z ≠ 0 the function f is equal to:
14. \[ f(z) = z^2 / |z| \]
15. \[ f(z) = |z|^2 / (1 - |z|) \]
16. \[ f(z) = 1 / |z| \]
17. \[ f(z) = (1 - |z|) / |z|^2 \]

18-22 Differentiation. Find the value of the derivative of
18. \[ f(z) = (z + i)(z + 1) \]
19. \[ f(z) = (z - 1)(z + 1) \]
20. \[ f(z) = z^2 / (1 - z)^2 \]
21. \[ f(z) = 1 / (z - i)^2 \]
22. \[ f(z) = z^2 / (z - i)^2 \]
23. \[ f(z) = z^2 / (z - i)^3 \]
24. TEAM PROJECT. Limit, Continuity, Derivative
(a) Limit. Prove that (1) is equivalent to the pair of relations
\[ \lim_{z \to a} \text{Re} f(z) = \text{Re} f(a), \lim_{z \to a} \text{Im} f(z) = \text{Im} f(a). \]
(b) Limit. If \( \lim f(z) \) exists, show that this limit is unique.
(c) Continuity. If \( z_1, z_2, \ldots \) are complex numbers for which \( \lim_{z \to z_0} f(z) = a \), and if \( f(z) \) is continuous at \( z = z_0 \), show that \( \lim_{z \to z_0} f(z) = a \).

25. WRITING PROJECT. Comparison with Calculus. Summarize the second part of this section beginning with Complex Function, and indicate what is conceptually analogous to calculus and what is not.


As we saw in the last section, to do complex analysis (i.e., "calculus in the complex") on any complex function, we require that function to be analytic on some domain that is differentiable in that domain.

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analytically of a complex function

\[ w = f(z) = u(x, y) + iv(x, y) \]

everywhere in D; here \( u_x = du_dx + dv_dy \) and \( u_y = du_dy - dv_dx \) (and similarly for v) are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorems 1 and 2.

Example: \( f(z) = z = x^2 - y^2 + 2i(x, y) \) is analytic for all \( z \) (see Example 3 in Sec. 13.3), and \( u_x = 2x \) and \( v_y = 2y \) satisfy (1), namely, \( u_x = 2t = u_y \) as well as \( u_y = -2y = -v_x \). More examples will follow.

Cauchy–Riemann Equations

Let \( f(z) = u(x, y) + iv(x, y) \) be defined and continuous in some neighborhood of a point \( z = x + iy \) and differentiable at \( z \) itself. Then, at that point, the first-order partial derivatives of \( u \) and \( v \) exist and satisfy the Cauchy–Riemann equations (1).

Hence, if \( f(z) \) is analytic in a domain \( D \), those partial derivatives exist and satisfy (1) at all points of \( D \).

4 The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897); see also Sec. 13.5) are the founders of complex analysis. Riemann received his Ph.D. in 1851 under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein’s theory of relativity; see Ref. [GenRef9] in App. 1.