where \( f^* \) is the odd periodic extension of \( f \) with the period \( 2L \) (Fig. 289). Since the initial deflection \( f(x) \) is continuous on the interval \( 0 \leq x \leq L \) and zero at the endpoints, it follows from (17) that \( u(x, t) \) is a continuous function of both variables \( x \) and \( t \) for all values of the variables. By differentiating (17) we see that \( u(x, t) \) is a solution of (1), provided \( f(x) \) is twice differentiable on the interval \( 0 < x < L \), and has one-sided second derivatives at \( x = 0 \) and \( x = L \), which are zero. Under these conditions \( u(x, t) \) is established as a solution of (1), satisfying (2) and (3) with \( g(x) = 0 \).

**Fig. 289.** Odd periodic extension of \( f(x) \)

**Generalized Solution.** If \( f'(x) \) and \( f''(x) \) are merely piecewise continuous (see Sec. 6.1), or if those one-sided derivatives are not zero, then for each \( t \) there will be finitely many values of \( x \) at which the second derivatives of \( u \) appearing in (1) do not exist. Except at these points the wave equation will still be satisfied. We may then regard \( u(x, t) \) as a "generalized solution," as it is called, that is, as a solution in a broader sense. For instance, a triangular initial deflection as in Example 1 (below) leads to a generalized solution.

**Physical Interpretation of the Solution (17).** The graph of \( f^*(x - ct) \) is obtained from the graph of \( f^*(x) \) by shifting the latter \( ct \) units to the right (Fig. 290). This means that \( f^*(x - ct)(c > 0) \) represents a wave that is traveling to the right as \( t \) increases. Similarly, \( f^*(x + ct) \) represents a wave that is traveling to the left, and \( u(x, t) \) is the superposition of these two waves.

**Fig. 290.** Interpretation of (17)

**Example 1**

**Vibrating String if the Initial Deflection Is Triangular**

Find the solution of the wave equation (1) satisfying (2) and corresponding to the triangular initial deflection

\[
f(x) = \begin{cases} 
\frac{2L}{L} & \text{if } 0 < x < \frac{L}{2} \\
\frac{2L}{L} (L - x) & \text{if } \frac{L}{2} < x < L 
\end{cases}
\]

and initial velocity zero. (Figure 291 shows \( f(x) = u(x, 0) \) at the top.)

**Solution.** Since \( f(x) = 0 \), we have \( \phi = 0 \) in (12), and from Example 4 in Sec. 11.3 we see that the \( B_k \) are given by (5). Sec. 11.3. Thus (12) takes the form

\[
u(x, t) = \frac{8L}{\pi^2} \sum \left\{ \sin \frac{\pi x}{L} \cos \frac{\pi c}{L} t - \frac{3}{\pi^2} \sin \frac{3\pi x}{L} \cos \frac{3\pi c}{L} t + \ldots \right\}
\]

For graphing the solution we may use \( u(x, 0) = f(x) \) and the above interpretation of the two functions in the representation (17). This leads to the graph shown in Fig. 291.

![Graph of the Solution](image)

**Fig. 291.** Solution \( u(x, t) \) in Example 1 for various values of \( t \) (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

**Problem Set 12.3**

1. **Frequency.** How does the frequency of the fundamental mode of the vibrating string depend on the length of the string? On the mass per unit length? What happens if we double the tension? Why is a contrabass larger than a violin?

2. **Physical Assumptions.** How would the motion of the string change if Assumption 3 were violated? Assumption 2? The second part of Assumption 1? The first part? Do we really need all these assumptions?

3. **String of length \( \pi \).** Write down the derivation in this section for length \( L = \pi \), to see the very substantial simplification of formulas in this case that may show ideas more clearly.

4. **CAS PROJECT.** Graphing Normal Modes. Write a program for graphing \( u_n \) with \( L = \pi \) and \( c^2 = 4 \) of your choice similarly as in Fig. 287. Apply the program to \( \phi_0, \phi_2, \phi_4 \). Also graph these solutions as surfaces over the \( x-t \)-plane. Explain the connection between these two kinds of graphs.

5-13 **DEFLECTION OF THE STRING**

Find \( u(x, t) \) for the string of length \( L = 1 \) and \( c^2 = 1 \) when the initial velocity is zero and the initial deflection with small \( k \) (say, 0.01) is as follows. Sketch or graph \( u(x, t) \) as in Fig. 291 in the text.

- \( k \) sin \( 3 \pi x \)
- \( 6. k \) (sin \( \pi x - \frac{3}{8} \) sin \( 3 \pi x \))
8. $ky(1 - x)$
9. $kx^2(1 - x)$
10. $\frac{1}{4}$
11. $\frac{1}{2}$
12. $\frac{1}{4}$
13. $2x - 4x^2$ if $0 < x < \frac{1}{4}$, $0$ if $\frac{1}{4} < x < 1$
14. Nonzero initial velocity. Find the deflection $u(x, t)$ of the string of length $L = \pi$ and $c^2 = 1$ for zero initial displacement and "triangular" initial velocity $u_t(x, 0) = 0.01 x$ if $0 \leq x \leq \frac{1}{4}$, $u_t(x, 0) = 0.01(\pi - x)$ if $\frac{1}{4} \leq x \leq \pi$. (Initial conditions with $u_t(x, 0) \neq 0$ are hard to realize experimentally.)

15. Substituting $u = F(x)G(t)$ into (21), show that
\[ F''(x)G(t) = \frac{\beta^2}{c^2} G(t) = \text{const} \]
\[ F(x) = A \cos \beta x + B \sin \beta x + C \cosh \beta x + D \sinh \beta x, \]
\[ G(t) = a t^2 e^{\beta t} + b \sin \beta t - c \cos \beta t. \]

16. Simply supported beam in Fig. 293A. Find solutions $u_n = F_n(x)G_n(t)$ of (21) corresponding to zero initial velocity and satisfying the boundary conditions (see Fig. 293A)
\[ u(x, 0) = 0, u(L, t) = 0 \]
and the initial condition
\[ u_t(x, 0) = 0, u_{x}(L, t) = 0 \]
(ends simply supported for all times $t$).

17. Find the solution of (21) that satisfies the conditions in Prob. 16 as well as the initial condition
\[ u(x, 0) = f(x) = x(L - x) \]
and the boundary conditions
\[ u_t(x, 0) = 0, u_{x}(L, t) = 0. \]

18. Compare the results of Probs. 17 and 7. What is the basic difference between the frequencies of the normal modes of the vibrating string and the vibrating beam?

19. Clamped beam in Fig. 293B. What are the boundary conditions for the clamped beam in Fig. 293B? Show that $F$ in Prob. 15 satisfies these conditions if $\beta L$ is a solution of the equation
\[ \cosh \beta L \cos \beta L = 1. \]
Determine approximate solutions of (22), for instance, graphically from the intersections of the curves of $\cos \beta L$ and $\frac{1}{2} \cosh \beta L$.

SEC. 12.4 D'Alembert's Solution of the Wave Equation. Characteristics

It is interesting that the solution (17), Sec. 12.3, of the wave equation
\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \]
can be immediately obtained by transforming (1) in a suitable way, namely, by introducing the new independent variables
\[ \tilde{v} = x + ct, \quad \tilde{w} = x - ct. \]

Then $u$ becomes a function of $\tilde{v}$ and $\tilde{w}$. The derivatives in (1) can now be expressed in terms of derivatives with respect to $\tilde{v}$ and $\tilde{w}$ by the use of the chain rule in Sec. 9.6. Denoting partial derivatives by subscripts, we see from (2) that $\tilde{v}_x = 1$ and $\tilde{w}_x = 1$. For simplicity let us denote $u(x, t)$, as a function of $\tilde{v}$ and $\tilde{w}$, by the same letter $u$. Then
\[ u_x = \tilde{u}_\tilde{v} \tilde{v}_x + \tilde{u}_\tilde{w} \tilde{w}_x = \tilde{u}_x + \tilde{u}_\tilde{w}. \]

We now apply the chain rule to the right side of this equation. We assume that all the partial derivatives involved are continuous, so that $\tilde{u}_{\tilde{v}_x} = \tilde{u}_{\tilde{w}_x}$. Since $\tilde{v}_x = 1$ and $\tilde{w}_x = 1$, we obtain
\[ \tilde{u}_{\tilde{v}_x} = (\tilde{u}_x + \tilde{u}_\tilde{w}) \tilde{v}_x + (\tilde{u}_x + \tilde{u}_\tilde{w}) \tilde{w}_x = \tilde{u}_x + 2\tilde{u}_{\tilde{v}_x} + \tilde{u}_{\tilde{w}_x}. \]

Transforming the other derivative in (1) by the same procedure, we find
\[ \tilde{u}_{\tilde{w}_x} = \tilde{u}_{\tilde{v}_x} - 2\tilde{u}_{\tilde{v}_x} + \tilde{u}_{\tilde{w}_x}. \]

By inserting these two results in (1) we get (see footnote 2 in App. A.3.2)
\[ \frac{\partial^2 u}{\partial \tilde{v} \partial \tilde{w}} = 0. \]

The point of the present method is that (3) can be readily solved by two successive integrations, first with respect to $\tilde{w}$ and then with respect to $\tilde{v}$. This gives
\[ \frac{\partial u}{\partial \tilde{v}} = h(v) \quad \text{and} \quad u = \int h(v) dv + \psi(w). \]
This shows that the expressions in the parentheses must be the Fourier coefficients $b_n$ of \( f(x) \); that is, by (4) in Sec. 11.3,

\[
b_n = A_n^b \sinh \frac{\pi b a}{a} = \frac{2}{\pi a} \int_0^a f(x) \sinh \frac{\pi x a}{a} \, dx.
\]

From this and (16) we see that the solution of our problem is

\[
a(x, y) = \sum_{n=1}^{\infty} a_n \sinh \frac{\pi y a}{a} \sin \frac{\pi x a}{a} + A_n^b \sinh \frac{\pi a}{a} \int_0^a f(x) \sin \frac{\pi x a}{a} \, dx.
\]

We have obtained this solution formally, neither considering convergence nor showing that the series for \( u, u_{xx}, \) and \( u_{yy} \) have the right sums. This can be proved if one assumes that \( f \) and \( f' \) are continuous and \( f'' \) is piecewise continuous on the interval \( 0 \leq x \leq a \). The proof is somewhat involved and relies on uniform convergence. It can be found in [C4] listed in App. 1.

### Unifying Power of Methods. Electrodynamics, Elasticity

The Laplace equation (14) also governs the electrostatic potential of electrical charges in any region that is free of these charges. Thus our steady-state heat problem can also be interpreted as an electrostatic potential problem. Then (17), (18) is the potential in the rectangular \( R \) when the upper side of \( R \) is at potential \( f(x) \) and the other three sides are grounded.

Actually, in the steady-state case, the two-dimensional wave equation (to be considered in Secs. 12.8, 12.9) also reduces to (14). Then (17), (18) is the displacement of a rectangular elastic membrane (rubber sheet, drumhead) that is fixed along its boundary, with three sides lying in the \( xy \)-plane and the fourth side given the displacement \( f(x) \).

This is another impressive demonstration of the unifying power of mathematics. It illustrates that entirely different physical systems may have the same mathematical model and can thus be treated by the same mathematical methods.

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**PROBLEM SET 12.6**

1. **Decay.** How does the rate of decay of (8) with fixed \( n \) depend on the specific heat, the density, and the thermal conductivity of the material?

2. **Decay.** If the first eigenfunction (8) of the bar decreases to half its value within 20 sec, what is the value of the diffusivity?

3. **Eigenvectors.** Sketch or graph and compare the first three eigenfunctions (8) with \( B_n = 1; c = 1 \), and \( L = \pi \) for \( t = 0, 0.1, 0.2, \ldots, 1.0 \).

4. **WAVE EQUATIONS.** Compare these PDEs with respect to general behavior of eigenfunctions and kind of boundary and initial conditions. State the difference between Fig. 291 in Sec. 12.3 and Fig. 295.

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**SEC. 12.6** Heat Equation: Solution by Fourier Series

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**8-7** LATERALLY INSULATED BAR

Find the temperature \( u(x, y) \) in a bar of silver of length 10 cm and constant cross section of area 1 cm\(^2\) (density 10.6 g/cm\(^3\), thermal conductivity 1.04 cal/deg cm sec), specific heat 0.056 cal/(g°C) that is perfectly insulated laterally, with ends kept at temperature 0°C and initial temperature \( f(x) \), where

5. \( f(x) = \sin 0.1 \pi x \)

6. \( f(x) = 4 - 0.8(x - 5) \)

7. \( f(x) = (10 - x) \)

8. **Arbitrary temperatures at ends.** If the ends \( x = 0 \) and \( x = L \) of the bar in the text are kept at constant temperatures \( u_1 \) and \( u_2 \) respectively, what is the temperature \( u(x) \) in the bar after a long time (theoretically, as \( t \to \infty \))? First guess, then calculate.

9. In Prob. 8 find the temperature at any time.

10. **Change of end temperatures.** Assume that the ends of the bar in Probs. 5-7 have been kept at 100°C for a long time. Then at some instant, call it \( t = 0 \), the temperature at \( x = L \) is suddenly changed to 0°C and kept at 0°C, whereas the temperature at \( x = 0 \) is kept at 100°C. Find the temperature in the middle of the bar at \( t = 1, 2, 3, 10, 50 \) sec. First guess, then calculate.

---

**BAR UNDER ADIABATIC CONDITIONS**

"Adiabatic" means no heat exchange with the neighborhood, because the bar is completely insulated, also at the ends. Physical Information: The heat flux at the ends is proportional to the value of \( u(x) \) there.

11. Show that for the completely insulated bar, \( u(0, t) = 0, u(L, t) = 0, u(x, t) = f(x) \) and separation of variables gives the following solution, with \( A_n \) given by (2) in Sec. 11.5:

\[
u(x, t) = \sum_{n=1}^{\infty} A_n \sinh \frac{\pi n a}{a} \sin \frac{\pi x a}{a}.
\]

12. **Find the temperature in Prob. 11 with \( L = \pi, c = 1, \) and \( T = 1, 2, 3, 4, 5 \).**

13. **Find the temperature in Prob. 12 with \( L = \pi, c = 1, \) and \( T = 1, 2, 3, 4, 5 \).**

14. **Find the temperature in Prob. 13 with \( L = \pi, c = 1, \) and \( T = 1, 2, 3, 4, 5 \).**

15. **Find the temperature in Prob. 14 with \( L = \pi, c = 1, \) and \( T = 1, 2, 3, 4, 5 \).**

16. A bar with heat generation of constant rate \( H \) (where \( H > 0 \)) is modeled by \( u_t = u_{xx} + H \). Solve this problem if \( L = \pi \) and the ends of the bar are kept at 0°C. Hint: Set \( u = v - H(t - x^2)/2 \).

17. **Heat flux.** The heat flux of a solution \( u(x, t) \) across \( x = 0 \) is defined by \( \phi(t) = -K_a u_t(0, t) \). Find \( \phi(t) \) for the solution (9). Explain the name. Is it physically understandable that \( \phi \) goes to 0 as \( t \to \infty \)?

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**18-25** TWO-DIMENSIONAL PROBLEMS

18. Laplace equation. Find the potential in the rectangular \( 0 \leq x \leq 20, 0 \leq y \leq 40 \) whose upper side is kept at potential 110 V and whose other sides are grounded.

19. **Find the potential in the square \( 0 \leq x \leq 20, 0 \leq y \leq 20 \) if the upper side is kept at the potential 1000 sin \( \frac{\pi}{10} x \) and the other sides are grounded.

20. **CAS PROJECT. Isotherms.** Find the steady-state solutions (temperatures) in the square plate in Fig. 297 with \( \alpha = 2 \) satisfying the following boundary conditions. Graph isotherms.

(a) \( u = 80 \sin \pi x \) on the upper side, 0 on the others.

(b) \( u = 0 \) on the vertical sides, assuming that the other sides are perfectly insulated.

(c) **Boundary conditions of your choice (such that the solution is not initially zero).**

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**Fig. 297. Square plate**