

PROBLEM SET 11.9

- 1. Review in complex.** Show that $1/i = -i$, $e^{-ix} = \cos x - i \sin x$, $e^{ix} + e^{-ix} = 2 \cos x$, $e^{ix} - e^{-ix} = 2i \sin x$, $e^{ikx} = \cos kx + i \sin kx$.

12-17 USE OF TABLE III IN SEC. 11.10. OTHER METHODS

- 12.** Find $\mathcal{F}\{f(x)\}$ for $f(x) = xe^{-x}$ if $x > 0$, $f(x) = 0$ if $x < 0$, by (9) in the text and formula 5 in Table III (with $a = 1$). *Hint.* Consider xe^{-x} and e^{-x} .
- 13.** Obtain $\mathcal{F}\{e^{-x^2/2}\}$ from Table III.
- 14.** In Table III obtain formula 7 from formula 8.
- 15.** In Table III obtain formula 1 from formula 2.
- 16. TEAM PROJECT. Shifting (a)** Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$.
(b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.
(c) Shifting on the w -Axis. Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.
- (d)** Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.
- 17.** What could give you the idea to solve Prob. 11 by using the solution of Prob. 9 and formula (9) in the text? Would this work?

18-25 DISCRETE FOURIER TRANSFORM

- 18.** Verify the calculations in Example 4 of the text.
- 19.** Find the transform of a general signal $f = [f_1 \ f_2 \ f_3 \ f_4]^T$ of four values.
 Find the inverse matrix in Example 4 of the text and use it to recover the given signal.
- 21.** Find the transform (the frequency spectrum) of a general signal of two values $[f_1 \ f_2]^T$.
- 22.** Recreate the given signal in Prob. 21 from the frequency spectrum obtained.
- 23.** Show that for a signal of eight sample values, $w = e^{-i/4} = (1 - i)/\sqrt{2}$. Check by squaring.
- 24.** Write the Fourier matrix \mathbf{F} for a sample of eight values explicitly.
- 25. CAS Problem.** Calculate the inverse of the 8×8 Fourier matrix. Transform a general sample of eight values and transform it back to the given data.

2-11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

2. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} e^x & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = e^{-|x|} \quad (-\infty < x < \infty)$

7. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

9. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

11. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

For $N = 2^p$ this breakdown can be repeated $p - 1$ times in order to finally arrive at $N/2$ problems of size 2 each, so that the number of multiplications is reduced as indicated above.
 We show the reduction from $N = 4$ to $M = N/2 = 2$ and then prove (22).

EXAMPLE 5 Fast Fourier Transform (FFT). Sample of $N = 4$ Values

When $N = 4$, then $w = w_N = -i$ as in Example 4 and $M = N/2 = 2$, hence $w = w_M = e^{-2\pi i/2} = e^{-\pi i} = -1$. Consequently,

$$\begin{aligned} \hat{\mathbf{f}}_{\text{ev}} &= \begin{bmatrix} \hat{f}_0 \\ \hat{f}_2 \end{bmatrix} = \mathbf{F}_2 \mathbf{f}_{\text{ev}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix} \\ \hat{\mathbf{f}}_{\text{od}} &= \begin{bmatrix} \hat{f}_1 \\ \hat{f}_3 \end{bmatrix} = \mathbf{F}_2 \mathbf{f}_{\text{od}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix} \end{aligned}$$

From this and (22a) we obtain

$$\begin{aligned} \hat{f}_0 &= \hat{f}_{\text{ev},0} + w_N^0 \hat{f}_{\text{od},0} = (f_0 + f_2) + (f_1 + f_3) = f_0 + f_1 + f_2 + f_3 \\ \hat{f}_1 &= \hat{f}_{\text{ev},1} + w_N^1 \hat{f}_{\text{od},1} = (f_0 - f_2) - i(f_1 + f_3) = f_0 - if_1 - f_2 + if_3 \end{aligned}$$

Similarly, by (22b),

$$\begin{aligned} \hat{f}_2 &= \hat{f}_{\text{ev},0} - w_N^0 \hat{f}_{\text{od},0} = (f_0 + f_2) - (f_1 + f_3) = f_0 - f_1 + f_2 - f_3 \\ \hat{f}_3 &= \hat{f}_{\text{ev},1} - w_N^1 \hat{f}_{\text{od},1} = (f_0 - f_2) - (-i)(f_1 - f_3) = f_0 + if_1 - f_2 - if_3 \end{aligned}$$

This agrees with Example 4, as can be seen by replacing 0, 1, 4, 9 with f_0, f_1, f_2, f_3 .

We prove (22). From (18) and (19) we have for the components of the DFT

$$\hat{f}_n = \sum_{k=0}^{N-1} w_N^{kn} f_k$$

Splitting into two sums of $M = N/2$ terms each gives

$$\hat{f}_n = \sum_{k=0}^{M-1} w_N^{2kn} f_{2k} + \sum_{k=0}^{M-1} w_N^{(2k+1)n} f_{2k+1}$$

We now use $w_N^2 = w_M$ and pull out w_N^n from under the second sum, obtaining

$$(23) \quad \hat{f}_n = \sum_{k=0}^{M-1} w_M^{kn} f_{\text{ev},k} + w_N^n \sum_{k=0}^{M-1} w_M^{kn} f_{\text{od},k}$$

The two sums are $f_{\text{ev},n}$ and $f_{\text{od},n}$, the components of the “half-size” transforms \mathbf{Ff}_{ev} and \mathbf{Ff}_{od} .

Formula (22a) is the same as (23). In (22b) we have $n + M$ instead of n . This causes a sign change in (23), namely $-w_N^n$ before the second sum because

$$w_N^M = e^{-2\pi i M/N} = e^{-2\pi i/2} = e^{-\pi i} = -1.$$

This gives the minus in (22b) and completes the proof.

The simple proof of this important theorem is quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 2–13 proceeds as for ODEs. Problems 16–23 concern PDEs solvable like ODEs. To help the student with them, we consider two typical examples.

EXAMPLE 2 Solving $u_{xx} - u = 0$ Like an ODE

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y .

Solution. Since no y -derivatives occur, we can solve this PDE like $u'' - u = 0$. In Sec. 2.2 we would have obtained $u = Ae^{x^2} + Be^{-x^2}$ with constant A and B . Here A and B may be functions of y , so that the answer is

$$u(x, y) = A(y)e^{x^2} + B(y)e^{-x^2}$$

with arbitrary functions A and B . We thus have a great variety of solutions. Check the result by differentiation. ■

EXAMPLE 3 Solving $u_{xy} = -u_x$ Like an ODE

Find solutions $u = u(x, y)$ of this PDE.

Solution. Setting $u_x = p$, we have $p_y = -p$, $p_y/p = -1$, $\ln |p| = -y + \tilde{c}(x)$, $p = c(x)e^{-y}$ and by integration with respect to x ,

$$u(x, y) = f(x)e^{-y} + g(y) \quad \text{where} \quad f(x) = \int c(x) dx,$$

here, $f(x)$ and $g(y)$ are arbitrary. ■

PROBLEM SET 12.1

Fundamental theorem. Prove it for second-order PDEs in two and three independent variables. *Hint.* Prove it by substitution.

13 VERIFICATION OF SOLUTIONS

Verify (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

5 Wave Equation (1) with suitable c

- $u = x^2 + t^2$
- $u = \cos 4t \sin 2x$
- $u = \sin kct \cos kx$
- $u = \sin at \sin bx$

9 Heat Equation (2) with suitable c

- $u = e^{-t} \sin x$
- $u = e^{-\omega^2 c^2 t} \cos \omega x$
- $u = e^{-9t} \sin \omega x$
- $u = e^{-\pi^2 t} \cos 25x$

0-13 Laplace Equation (3)

- $u = e^x \cos y, e^x \sin y$
- $u = \arctan (y/x)$
- $u = \cos y \sinh x, \sin y \cosh x$

- (3) and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

16-23 PDES SOLVABLE AS ODES

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form), so that the other variable(s) can be treated as parameter(s). Solve for $u = u(x, y)$:

- 16. $u_{yy} = 0$
- 17. $u_{xx} + 16\pi^2 u = 0$
- 18. $25u_{yy} - 4u = 0$
- 19. $u_y + y^2 u = 0$
- 20. $2u_{xx} + 9u_x + 4u = -3 \cos x - 29 \sin x$
- 21. $u_{yy} + 6u_y + 13u = 4e^{3y}$
- 22. $u_{xy} = u_x$
- 23. $x^2 u_{xx} + 2xu_x - 2u = 0$

24. Surface of revolution. Show that the solutions $z = z(x, y)$ of $yz_x = xz_y$ represent surfaces of revolution. Give examples. *Hint.* Use polar coordinates r, θ and show that the equation becomes $z_\theta = 0$.

25. System of PDEs. Solve $u_{xx} = 0, u_{yy} = 0$

12.2 Modeling: Vibrating String, Wave Equation

In this section we model a vibrating string, which will lead to our first important PDE, that is, equation (3) which will then be solved in Sec. 12.3. *The student should pay very close attention to this delicate modeling process and detailed derivation starting from scratch*, as the skills learned can be applied to modeling other phenomena in general and in particular to modeling a vibrating membrane (Sec. 12.7).

We want to derive the PDE modeling small transverse vibrations of an elastic string, such as a violin string. We place the string along the x -axis, stretch it to length L , and fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$; see Fig. 286.

$u(x, t)$ will be the solution of a PDE that is the model of our physical system to be derived. This PDE should not be too complicated, so that we can solve it. Reasonable simplifying assumptions (just as for ODEs modeling vibrations in Chap. 2) are as follows.

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well.

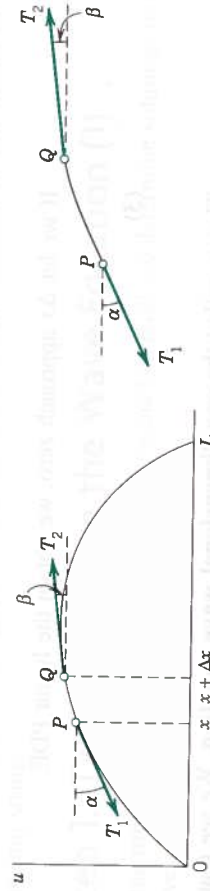


Fig. 286. Deflected string at fixed time t . Explanation on p. 544

where f^* is the odd periodic extension of f with the period $2L$ (Fig. 289). Since the initial deflection $f(x)$ is continuous on the interval $0 \leq x \leq L$ and zero at the endpoints, it follows from (17) that $u(x, t)$ is a continuous function of both variables x and t for all values of the variables. By differentiating (17) we see that $u(x, t)$ is a solution of (1), provided $f(x)$ is twice differentiable on the interval $0 < x < L$, and has one-sided second derivatives at $x = 0$ and $x = L$, which are zero. Under these conditions $u(x, t)$ is established as a solution of (1), satisfying (2) and (3) with $g(x) \equiv 0$.



Fig. 289. Odd periodic extension of $f(x)$

Generalized Solution. If $f'(x)$ and $f''(x)$ are merely piecewise continuous (see Sec. 6.1), or if those one-sided derivatives are not zero, then for each t there will be finitely many values of x at which the second derivatives of u appearing in (1) do not exist. Except at these points the wave equation will still be satisfied. We may then regard $u(x, t)$ as a “generalized solution,” as it is called, that is, as a solution in a broader sense. For instance, a triangular initial deflection as in Example 1 (below) leads to a generalized solution.

Physical Interpretation of the Solution (17). The graph of $f^*(x - ct)$ is obtained from the graph of $f^*(x)$ by shifting the latter ct units to the right (Fig. 290). This means that $f^*(x - ct)(c > 0)$ represents a wave that is traveling to the right as t increases. Similarly, $f^*(x + ct)$ represents a wave that is traveling to the left, and $u(x, t)$ is the superposition of these two waves.

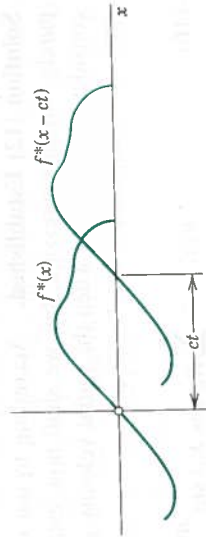


Fig. 290. Interpretation of (17)

EXAMPLE 1 Vibrating String if the Initial Deflection Is Triangular

Find the solution of the wave equation (1) satisfying (2) and corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 291 shows $f(x) = u(x, 0)$ at the top.)

Solution. Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and from Example 4 in Sec. 11.3 we see that the B_n are given by (5), Sec. 11.3. Thus (12) takes the form

$$u(x, t) = \frac{8k}{\pi^2} \left[\frac{1}{1^2} \sin \frac{\pi}{L} x \cos \frac{\pi c}{L} t - \frac{1}{3^2} \sin \frac{3\pi}{L} x \cos \frac{3\pi c}{L} t + \dots \right]$$

For graphing the solution we may use $u(x, 0) = f(x)$ and the above interpretation of the two functions in the representation (17). This leads to the graph shown in Fig. 291.

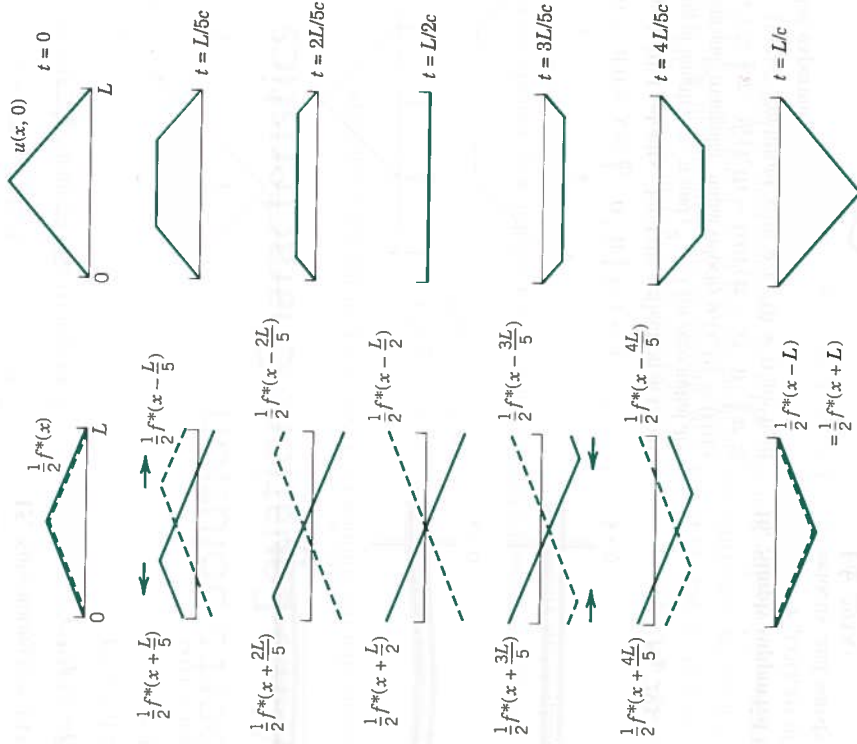


Fig. 291. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

PROBLEM SET 12.3

- Frequency.** How does the frequency of the fundamental mode of the vibrating string depend on the length of the string? On the mass per unit length? What happens if we double the tension? Why is a contrabass larger than a violin?
- Physical Assumptions.** How would the motion of the string change if Assumption 3 were violated? Assumption 2? The second part of Assumption 1? The first part? Do we really need all these assumptions?
- String of length π .** Write down the derivation in this section for length $L = \pi$, to see the very substantial simplification of formulas in this case that may show ideas more clearly.

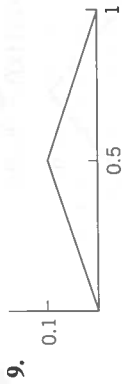
4. CAS PROJECT. Graphing Normal Modes. Write a program for graphing u_n with $L = \pi$ and c^2 of your choice similarly as in Fig. 287. Apply the program to u_2, u_3, u_4 . Also graph these solutions as surfaces over the xt -plane. Explain the connection between these two kinds of graphs.

5-13 DEFLECTION OF THE STRING

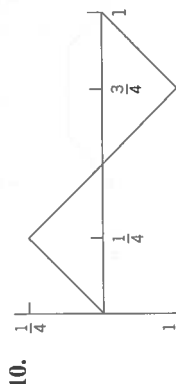
Find $u(x, t)$ for the string of length $L = 1$ and $c^2 = 1$ when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows. Sketch or graph $u(x, t)$ as in Fig. 291 in the text.

- $k \sin 3\pi x$
- $k(\sin \pi x - \frac{1}{2} \sin 2\pi x)$

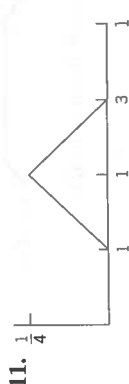
7. $kx(1-x)$



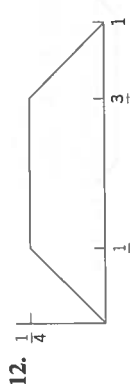
10.



11.



12.



13. $2x - 4x^2$ if $0 < x < \frac{1}{2}$, 0 if $\frac{1}{2} < x < 1$

14. **Nonzero initial velocity.** Find the deflection $u(x, t)$ of the string of length $L = \pi$ and $c^2 = 1$ for zero initial displacement and "triangular" initial velocity $u_t(x, 0) = 0.01x$ if $0 \leq x \leq \frac{1}{2}\pi$, $u_t(x, 0) = 0.01(\pi - x)$ if $\frac{1}{2}\pi \leq x \leq \pi$. (Initial conditions with $u_t(x, 0) \neq 0$ are hard to realize experimentally.)

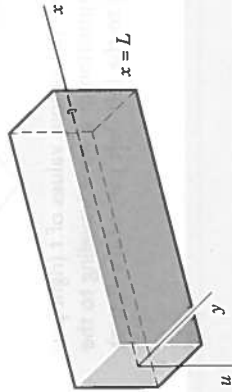


Fig. 292. Elastic beam

15-20 SEPARATION OF A FOURTH-ORDER PDE. VIBRATING BEAM

By the principles used in modeling the string it can be shown that small free vertical vibrations of a uniform elastic beam (Fig. 292) are modeled by the fourth-order PDE

$$(21) \quad \frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \quad (\text{Ref. [C11]})$$

where $c^2 = EI/\rho A$ ($E =$ Young's modulus of elasticity, $I =$ moment of inertia of the cross section with respect to the

y -axis in the figure, $\rho =$ density, $A =$ cross-sectional area). (Bending of a beam under a load is discussed in Sec. 3.3.)

15. Substituting $u = F(x)G(t)$ into (21), show that

$$F^{(4)}/F = -\ddot{G}/c^2 G = \beta^4 = \text{const.}$$

$$F(x) = A \cos \beta x + B \sin \beta x + C \cosh \beta x + D \sinh \beta x,$$

$$G(t) = a \cos c\beta^2 t + b \sin c\beta^2 t.$$

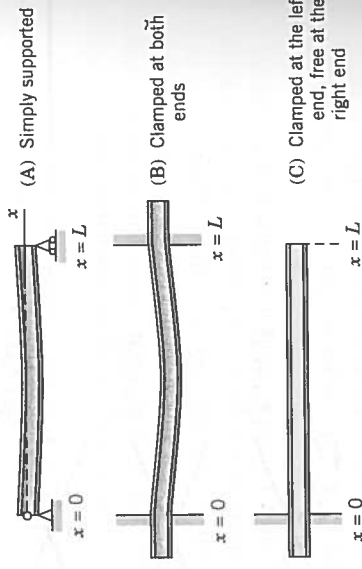


Fig. 293. Supports of a beam

16. **Simply supported beam in Fig. 293A.** Find solutions $u_n = F_n(x)G_n(t)$ of (21) corresponding to zero initial velocity and satisfying the boundary conditions (see Fig. 293A)

$$u(0, t) = 0, \quad u(L, t) = 0$$

(ends simply supported for all times t),

$$u_{xx}(0, t) = 0, \quad u_{xx}(L, t) = 0$$

(zero moments, hence zero curvature, at the ends).

17. Find the solution of (21) that satisfies the conditions in Prob. 16 as well as the initial condition

$$u(x, 0) = f(x) = x(L-x).$$

18. Compare the results of Probs. 17 and 7. What is the basic difference between the frequencies of the normal modes of the vibrating string and the vibrating beam?

19. **Clamped beam in Fig. 293B.** What are the boundary conditions for the clamped beam in Fig. 293B? Show that F in Prob. 15 satisfies these conditions if βL is a solution of the equation

$$(22) \quad \cosh \beta L \cos \beta L = 1.$$

Determine approximate solutions of (22), for instance, graphically from the intersections of the curves of $\cos \beta L$ and $1/\cosh \beta L$.

20. **Clamped-free beam in Fig. 293C.** If the beam is clamped at the left and free at the right (Fig. 293C), the boundary conditions are

$$u(0, t) = 0, \quad u_x(0, t) = 0, \quad \cosh \beta L \cos \beta L = -1. \quad (23)$$

Find approximate solutions of (23).

12.4 D'Alembert's Solution of the Wave Equation. Characteristics

It is interesting that the solution (17), Sec. 12.3, of the wave equation

$$(1) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho},$$

can be immediately obtained by transforming (1) in a suitable way, namely, by introducing the new independent variables

$$(2) \quad v = x + ct, \quad w = x - ct.$$

Then u becomes a function of v and w . The derivatives in (1) can now be expressed in terms of derivatives with respect to v and w by the use of the chain rule in Sec. 9.6. Denoting partial derivatives by subscripts, we see from (2) that $v_x = 1$ and $w_x = 1$. For simplicity let us denote $u(x, t)$, as a function of v and w , by the same letter u . Then

$$u_x = u_v v_x + u_w w_x = u_v + u_w.$$

We now apply the chain rule to the right side of this equation. We assume that all the partial derivatives involved are continuous, so that $u_{vw} = u_{wv}$. Since $v_x = 1$ and $w_x = 1$, we obtain

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}.$$

Transforming the other derivative in (1) by the same procedure, we find

$$u_{tt} = c^2(u_{vv} - 2u_{vw} + u_{ww}).$$

By inserting these two results in (1) we get (see footnote 2 in App. A3.2)

$$(3) \quad u_{vw} \equiv \frac{\partial^2 u}{\partial v \partial w} = 0.$$

The point of the present method is that (3) can be readily solved by two successive integrations, first with respect to w and then with respect to v . This gives

$$\frac{\partial u}{\partial v} = h(v) \quad \text{and} \quad u = \int h(v) dv + \psi(w).$$