PROBLEM SET 11.1

1–5 PERIOD, FUNDAMENTAL PERIOD

The *fundamental period* is the smallest positive period. Find it for

1. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$
2. $\cos nx$, $\sin nx$, $\cos \frac{2\pi x}{k}$, $\sin \frac{2\pi x}{k}$, $\cos \frac{2\pi nx}{k}$, $\sin \frac{2\pi nx}{k}$
3. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ (a, b, constant) has the period p. Thus all functions of period p form a **vector space**.
4. **Change of scale.** If $f(x)$ has period p, show that $f(ax)$, $a \neq 0$, and $f(x/b)$, $b \neq 0$, are periodic functions of x of periods p/a and bp, respectively. Give examples.
5. Show that $f =$ const is periodic with any period but has no fundamental period.

6–10 GRAPHS OF 2π–PERIODIC FUNCTIONS

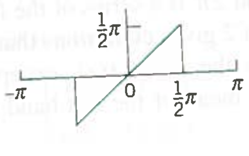
Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

6. $f(x) = |x|$
7. $f(x) = |\sin x|$, $f(x) = \sin |x|$
8. $f(x) = e^{-|x|}$, $f(x) = |e^{-x}|$
9. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$
10. $f(x) = \begin{cases} -\cos^2 x & \text{if } -\pi < x < 0 \\ \cos^2 x & \text{if } 0 < x < \pi \end{cases}$
11. **Calculus review.** Review integration techniques for integrals as they are likely to arise from the Euler formulas, for instance, definite integrals of $x \cos nx$, $x^2 \sin nx$, $e^{-2x} \cos nx$, etc.

12–21 FOURIER SERIES

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

12. $f(x)$ in Prob. 6
13. $f(x)$ in Prob. 9
14. $f(x) = x^2$ $(-\pi < x < \pi)$
15. $f(x) = x^2$ $(0 < x < 2\pi)$
16.



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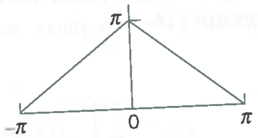

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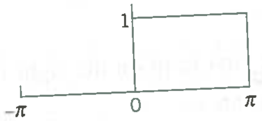

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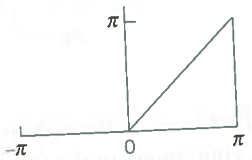

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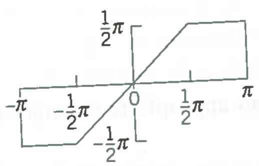

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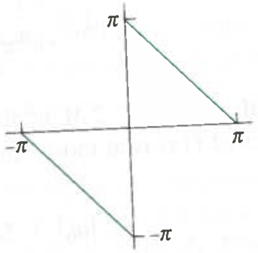


22. **CAS EXPERIMENT. Graphing.** Write a program for graphing partial sums of the following series. Guess from the graph what $f(x)$ the series may represent. Confirm or disprove your guess by using the Euler formulas.
   (a) $2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots)$
   $- 2(\frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x \cdots)$
   (b) $\frac{1}{2} + \frac{4}{\pi^2}\left(\cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \cdots\right)$
   (c) $\frac{2}{3}\pi^2 + 4(\cos x - \frac{1}{4} \cos 2x + \frac{1}{9} \cos 3x - \frac{1}{16} \cos 4x + - \cdots)$
23. **Discontinuities.** Verify the last statement in Theorem 2 for the discontinuities of $f(x)$ in Prob. 21.
24. **CAS EXPERIMENT. Orthogonality.** Integrate and graph the integral of the product $\cos mx \cos nx$ (with various integer m and n of your choice) from $-a$ to a as a function of a and conclude orthogonality of $\cos mx$ and $\cos nx$ $(m \neq n)$ for $a = \pi$ from the graph. For what m and n will you get orthogonality for $a = \pi/2, \pi/3, \pi/4$? Other a? Extend the experiment to $\cos mx \sin nx$ and $\sin mx \sin nx$.
25. **CAS EXPERIMENT. Order of Fourier Coefficients.** The order seems to be $1/n$ if f is discontinous, and $1/n^2$ if f is continuous but $f' = df/dx$ is discontinuous, $1/n^3$ if f and f' are continuous but f'' is discontinuous, etc. Try to verify this for examples. Try to prove it by integrating the Euler formulas by parts. What is the practical significance of this?

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

We now expand our initial basic discussion of Fourier series.

Orientation. This section concerns three topics:

1. Transition from period 2π to any period $2L$, for the function f, simply by a transformation of scale on the x-axis.
2. Simplifications. Only cosine terms if f is even ("Fourier cosine series"). Only sine terms if f is odd ("Fourier sine series").
3. Expansion of f given for $0 \leqq x \leqq L$ in two Fourier series, one having only cosine terms and the other only sine terms ("half-range expansions").

1. From Period 2π to Any Period $p = 2L$

Clearly, periodic functions in applications may have any period, not just 2π as in the last section (chosen to have simple formulas). The notation $p = 2L$ for the period is practical because L will be a length of a violin string in Sec. 12.2, of a rod in heat conduction in Sec. 12.5, and so on.

The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable v such that $f(x)$, as a function of v, has period 2π. If we set

$$(1) \qquad \text{(a)} \quad x = \frac{p}{2\pi} v, \qquad \text{so that} \qquad \text{(b)} \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x$$

then $v = \pm\pi$ corresponds to $x = \pm L$. This means that f, as a function of v, has period 2π and, therefore, a Fourier series of the form

$$(2) \qquad f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

with coefficients obtained from (6) in the last section

$$(3) \qquad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) dv, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \cos nv \, dv,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \sin nv \, dv.$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 272a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) ***Odd periodic extension.*** Similarly, from (6**) we obtain

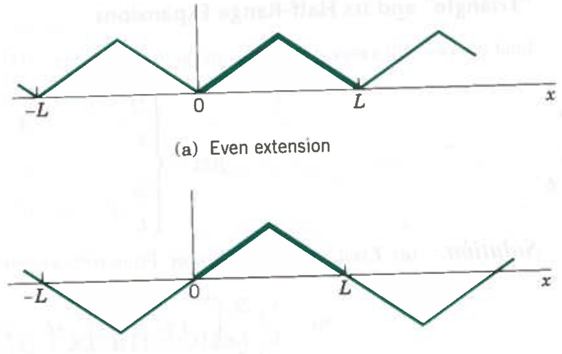
$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 272b)

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5. ■



(a) Even extension

(b) Odd extension

Fig. 272. Periodic extensions of $f(x)$ in Example 6

PROBLEM SET 11.2

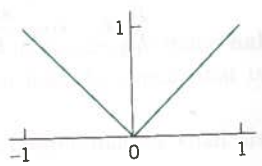
1–7 EVEN AND ODD FUNCTIONS

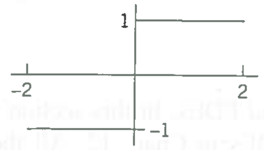
Are the following functions even or odd or neither even nor odd?

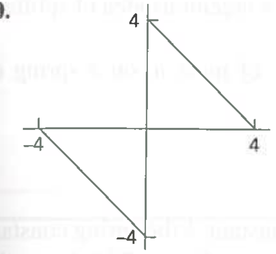
1. e^x, $e^{-|x|}$, $x^3\cos nx$, $x^2\tan\pi x$, $\sinh x - \cosh x$
2. $\sin^2 x$, $\sin(x^2)$, $\ln x$, $x/(x^2+1)$, $x\cot x$
3. Sums and products of even functions
4. Sums and products of odd functions
5. Absolute values of odd functions
6. Product of an odd times an even function
7. Find all functions that are both even and odd.

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

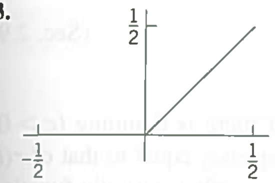
8. 

9. 

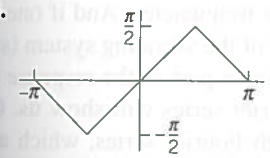
10. 

11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

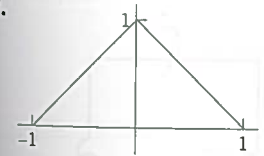
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$

13. 

14. $f(x) = \cos\pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$

15. 

16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$

17. 

18. **Rectifier.** Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0\cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.

19. **Trigonometric Identities.** Show that the familiar identities $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ and $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.

20. **Numeric Values.** Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.

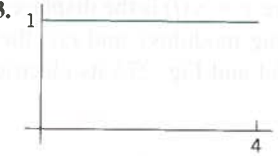
21. **CAS PROJECT. Fourier Series of 2L-Periodic Functions.** (a) Write a program for obtaining partial sums of a Fourier series (5).

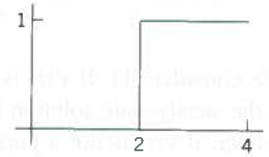
(b) Apply the program to Probs. 8–11, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.

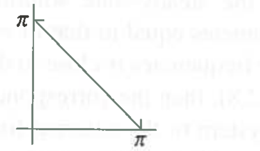
22. Obtain the Fourier series in Prob. 8 from that in Prob. 17.

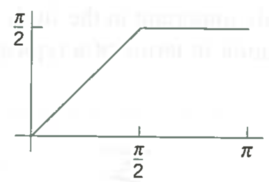
23–29 HALF-RANGE EXPANSIONS

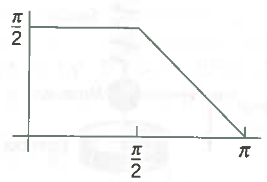
Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

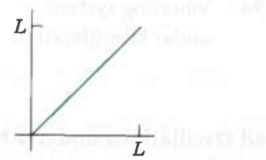
23. 

24. 

25. 

26. 

27. 

28. 

29. $f(x) = \sin x \quad (0 < x < \pi)$

30. Obtain the solution to Prob. 26 from that of Prob. 27.

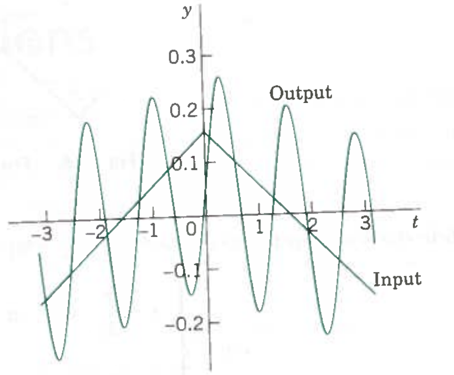


Fig. 277. Input and steady-state output in Example 1

PROBLEM SET 11.3

1. **Coefficients C_n.** Derive the formula for C_n from A_n and B_n.
2. **Change of spring and damping.** In Example 1, what happens to the amplitudes C_n if we take a stiffer spring, say, of $k = 49$? If we increase the damping?
3. **Phase shift.** Explain the role of the B_n's. What happens if we let $c \to 0$?
4. **Differentiation of input.** In Example 1, what happens if we replace $r(t)$ with its derivative, the rectangular wave? What is the ratio of the new C_n to the old ones?
5. **Sign of coefficients.** Some of the A_n in Example 1 are positive, some negative. All B_n are positive. Is this physically understandable?

6–11 GENERAL SOLUTION

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

6. $r(t) = \sin \alpha t + \sin \beta t,\ \omega^2 \neq \alpha^2, \beta^2$
7. $r(t) = \sin t,\ \omega = 0.5, 0.9, 1.1, 1.5, 10$
8. **Rectifier.** $r(t) = \pi/4\,|\cos t|$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t),\ |\omega| \neq 0, 2, 4, \cdots$
9. What kind of solution is excluded in Prob. 8 by $|\omega| \neq 0, 2, 4, \cdots$?
10. **Rectifier.** $r(t) = \pi/4\,|\sin t|$ if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t),\ |\omega| \neq 0, 2, 4, \cdots$
11. $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi, \end{cases} \quad |\omega| \neq 1, 3, 5, \cdots$
12. **CAS Program.** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 7 and 11 with initial values of your choice.

13–16 STEADY-STATE DAMPED OSCILLATIONS

Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details. In Probs. 14–16 sketch $r(t)$.

13. $r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$
14. $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$
15. $r(t) = t(\pi^2 - t^2)$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$
16. $r(t) = \begin{cases} t & \text{if } -\pi/2 < t < \pi/2 \\ \pi - t & \text{if } \pi/2 < t < 3\pi/2 \end{cases}$ and $r(t + 2\pi) = r(t)$

17–19 *RLC*-CIRCUIT

Find the steady-state current $I(t)$ in the *RLC*-circuit in Fig. 275, where $R = 10\ \Omega$, $L = 1$ H, $C = 10^{-1}$ F and with $E(t)$ V as follows and periodic with period 2π. Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. *Hint.* Remember that the ODE contains $E'(t)$, not $E(t)$, cf. Sec. 2.9.

17. $E(t) = \begin{cases} -50t^2 & \text{if } -\pi < t < 0 \\ 50t^2 & \text{if } 0 < t < \pi \end{cases}$
18. $E(t) = \begin{cases} 100(t - t^2) & \text{if } -\pi < t < 0 \\ 100(t + t^2) & \text{if } 0 < t < \pi \end{cases}$
19. $E(t) = 200t(\pi^2 - t^2) \quad (-\pi < t < \pi)$
20. **CAS EXPERIMENT. Maximum Output Term.** Graph and discuss outputs of $y'' + cy' + ky = r(t)$ with $r(t)$ as in Example 1 for various c and k with emphasis on the maximum C_n and its ratio to the second largest $|C_n|$.

11.4 Approximation by Trigonometric Polynomials

Fourier series play a prominent role not only in differential equations but also in **approximation theory**, an area that is concerned with approximating functions by other functions—usually simpler functions. Here is how Fourier series come into the picture.

Let $f(x)$ be a function on the interval $-\pi \le x \le \pi$ that can be represented on this interval by a Fourier series. Then the ***N*th partial sum** of the Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \tag{1}$$

is an approximation of the given $f(x)$. In (1) we choose an arbitrary N and keep it fixed. Then we ask whether (1) is the "best" approximation of f by a **trigonometric polynomial of the same degree N**, that is, by a function of the form

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \qquad (N \text{ fixed}). \tag{2}$$

Here, "best" means that the "error" of the approximation is as small as possible.

Of course we must first define what we mean by the **error** of such an approximation. We could choose the maximum of $|f(x) - F(x)|$. But in connection with Fourier series it is better to choose a definition of error that measures the goodness of agreement between f and F *on the whole interval* $-\pi \le x \le \pi$. This is preferable since the sum f of a Fourier series may have jumps: F in Fig. 278 is a good overall approximation of f, but the maximum of $|f(x) - F(x)|$ (more precisely, the *supremum*) is large. We choose

$$E = \int_{-\pi}^{\pi} (f - F)^2\, dx. \tag{3}$$

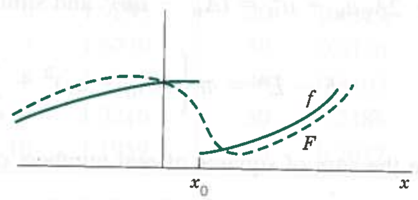


Fig. 278. Error of approximation